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Light-Tailed Asymptotics of Stationary Tail Probability Vectors of Markov Chains of M/G/1 Type*

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ABSTRACT

□ *This paper studies the light-tailed asymptotics of the stationary tail probability vectors of a Markov chain of M/G/1 type. Almost all related studies have focused on the typical case, where the transition block matrices in the non-boundary levels have a dominant impact on the decay rate of the stationary tail probability vectors and their decay is aperiodic. In this paper, we study not only the typical case but also atypical cases such that the stationary tail probability vectors decay periodically and/or their decay rate is determined by the tail distribution of jump sizes from the boundary level. We derive light-tailed asymptotic formulae for the stationary tail probability vectors by locating the dominant poles of the generating function of the sequence of those vectors. Further we discuss the positivity of the dominant terms of the obtained asymptotic formulae.*

Keywords Light-tailed asymptotics; Markov chain of M/G/1 type; Stationary tail probability vector; generating function; Dominant pole.

Mathematics Subject Classification Primary 60K25; Secondary 60J10.

1 Introduction

This paper considers a Markov chain $\{(X_n, S_n);\ n = 0, 1, \dots\}$ of M/G/1 type [20], where $X_n \in \{0, 1, \dots\}$ and

$$\begin{aligned} &S_n \in \mathbb{M}_0 \triangleq \{1, 2, \dots, M_0\}, && \text{if } X_n = 0, \\ &S_n \in \mathbb{M} \triangleq \{1, 2, \dots, M\}, && \text{otherwise.} \end{aligned}$$

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The sets of states $\{(0,j); j \in \mathbb{M}_0\}$ and $\{(k,j); j \in \mathbb{M}\}$ ($k = 1,2,\dots$) are called level 0 and level k, respectively. Arranging the states in lexicographical order, the transition probability matrix $\boldsymbol{T}$ of $\{(X_n, S_n);\ n = 0,1,\dots\}$ is given by

$$\boldsymbol{T} = \begin{array}{c} \\ \text{lev. } 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \text{lev. } 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{pmatrix} \boldsymbol{B}(0) & \boldsymbol{B}(1) & \boldsymbol{B}(2) & \boldsymbol{B}(3) & \cdots \\ \boldsymbol{C}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \boldsymbol{A}(3) & \cdots \\ \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \boldsymbol{A}(2) & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}(0) & \boldsymbol{A}(1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}, \tag{1}$$

where $\boldsymbol{A}(k)$ ($k = 0,1,\dots$) is an $M \times M$ matrix, $\boldsymbol{B}(0)$ is an $M_0 \times M_0$ matrix, $\boldsymbol{B}(k)$ ($k = 1,2,\dots$) is an $M_0 \times M$ matrix, and $\boldsymbol{C}(0)$ is an $M \times M_0$ matrix.

We define $\boldsymbol{A}$ and $\boldsymbol{B}$ as

$$\boldsymbol{A} = \sum_{k=0}^{\infty} \boldsymbol{A}(k), \qquad \boldsymbol{B} = \sum_{k=1}^{\infty} \boldsymbol{B}(k),$$

respectively. We assume that $\boldsymbol{A}$ is a stochastic matrix and $\boldsymbol{B}(0)\boldsymbol{e} + \boldsymbol{B}\boldsymbol{e} = \boldsymbol{e}$, where $\boldsymbol{e}$ denotes a column vector of ones with an appropriate dimension. Let $\boldsymbol{\pi}$ denote a $1 \times M$ vector such that $\boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$. Note here that if $\boldsymbol{A}$ is irreducible, $\boldsymbol{\pi}$ is uniquely determined.

Throughout this paper, we make the following assumption.

Assumption 1.1

(a) $\boldsymbol{T}$ is irreducible.

(b) $\boldsymbol{A}$ is irreducible.

(c) $\rho \triangleq \boldsymbol{\pi}\boldsymbol{\beta}_A < 1$, where $\boldsymbol{\beta}_A = \sum_{k=1}^{\infty} k\boldsymbol{A}(k)\boldsymbol{e}$.

(d) $\boldsymbol{\beta}_B \triangleq \sum_{k=1}^{\infty} k\boldsymbol{B}(k)\boldsymbol{e}$ is a finite vector.

Let $\boldsymbol{x}$ denote the stationary probability vector of $\boldsymbol{T}$, i.e., $\boldsymbol{x}\boldsymbol{T} = \boldsymbol{x}$ and $\boldsymbol{x}\boldsymbol{e} = 1$. It is known that under Assumption 1.1, $\{(X_n, S_n);\ n = 0,1,\dots\}$ is irreducible and positive recurrent (see Proposition 3.1 in Chapter XI of [4]). Therefore if Assumption 1.1 holds, then $\boldsymbol{x} > \boldsymbol{0}$, which is uniquely determined. Let $\boldsymbol{x}(k)$ ($k = 0,1,\dots$) denote a subvector of $\boldsymbol{x}$ corresponding to level k. We then have $\boldsymbol{x} = (\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \dots)$ and

$$\boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{B}(k) + \sum_{l=1}^{k+1} \boldsymbol{x}(l)\boldsymbol{A}(k+1-l), \qquad k = 1,2,\dots. \tag{2}$$

Further let $\overline{\boldsymbol{x}}(k) = \sum_{l=k+1}^{\infty} \boldsymbol{x}(l)$ ($k = 0, 1, \dots$), which is a positive vector. We call $\overline{\boldsymbol{x}}(k)$'s stationary tail probability vectors of $\boldsymbol{T}$ hereafter.

In this paper, we study the light-tailed asymptotics of $\{\overline{\boldsymbol{x}}(k); k = 0, 1, \dots\}$. The following is the definition of a light-tailed sequence of nonnegative matrices (including vectors).

Definition 1.1 Let $\boldsymbol{Y}(k)$'s ($k = 0, 1, \dots$) denote nonnegative matrices with the same dimension, and let $\boldsymbol{Y}^*(z)$ denote the generating function of $\{\boldsymbol{Y}(k)\}$ defined by the power series $\sum_{k=0}^{\infty} z^k \boldsymbol{Y}(k)$, whose convergence radius is given by $\sup\{y > 0; \sum_{k=0}^{\infty} y^k \boldsymbol{Y}(k) < \infty\}$. The sequence $\{\boldsymbol{Y}(k)\}$ is said to be light-tailed if the convergence radius of $\boldsymbol{Y}^*(z)$ is greater than one.

Remark 1.1 Let $\overline{\boldsymbol{Y}}(k) = \sum_{l=k+1}^{\infty} \boldsymbol{Y}(l)$ for $k = 0, 1, \dots$, and let $\overline{\boldsymbol{Y}}^*(z)$ denote the generating function of $\{\overline{\boldsymbol{Y}}(k)\}$ defined by $\sum_{k=0}^{\infty} z^k \overline{\boldsymbol{Y}}(k)$. We then have

$$\overline{\boldsymbol{Y}}^*(z) = \frac{\boldsymbol{Y}^*(1) - \boldsymbol{Y}^*(z)}{1 - z},$$

which implies that $\{\overline{\boldsymbol{Y}}(k)\}$ is light-tailed if and only if $\{\boldsymbol{Y}(k)\}$ is light-tailed.

Let $\overline{\boldsymbol{x}}^*(z)$ and $\boldsymbol{x}^*(z)$ denote the generating functions of $\{\overline{\boldsymbol{x}}(k); k = 0, 1, \dots\}$ and $\{\boldsymbol{x}(k); k = 1, 2, \dots\}$, respectively. We then have

$$\overline{\boldsymbol{x}}^*(z) = \frac{\boldsymbol{x}^*(1) - \boldsymbol{x}^*(z)}{1 - z}. \tag{3}$$

It also follows from (2) that

$$\boldsymbol{x}^*(z)(\boldsymbol{I} - \boldsymbol{A}^*(z)/z) = (\boldsymbol{x}(0)\boldsymbol{B}^*(z) - \boldsymbol{x}(1)\boldsymbol{A}(0)), \tag{4}$$

where $\boldsymbol{A}^*(z)$ and $\boldsymbol{B}^*(z)$ denote the generating functions of $\{\boldsymbol{A}(k)\}$ and $\{\boldsymbol{B}(k)\}$ defined by $\sum_{k=0}^{\infty} z^k \boldsymbol{A}(k)$ and $\sum_{k=1}^{\infty} z^k \boldsymbol{B}(k)$, respectively.

Let r_A and r_B denote the convergence radii of $\boldsymbol{A}^*(z)$ and $\boldsymbol{B}^*(z)$, respectively. We then make the second assumption.

Assumption 1.2 (a) $r_A > 1$, and (b) $r_B > 1$.

Proposition 1.1 below follows from Remark 1.1, Theorem 3.1 in [15] and Theorem 2 in [16].

Proposition 1.1 *Under Assumption 1.1, $\{\overline{\boldsymbol{x}}(k)\}$ is light-tailed if and only if Assumption 1.2 holds.*

For further discussion, we introduce the following definition.

Definition 1.2 For any finite square matrix $\boldsymbol{X}$ with (possibly) complex number elements, let $\delta(\boldsymbol{X})$ denote a maximum-modulus eigenvalue of $\boldsymbol{X}$, whose argument $\arg\delta(\boldsymbol{X})$ is nonnegative, and whose real part $\mathrm{Re}\,\delta(\boldsymbol{X})$ is not less than those of the other eigenvalues of maximum modulus. Clearly, $|\delta(\boldsymbol{X})|$ is the spectral radius of $\boldsymbol{X}$. In addition, if $\boldsymbol{X}$ is nonnegative and irreducible, $\delta(\boldsymbol{X}) > 0$ is the Perron-Frobenius eigenvalue of $\boldsymbol{X}$.

We now make the third assumption, Assumption 1.3 below. Unless otherwise stated, Assumptions 1.1, 1.2 and 1.3 are valid throughout this paper. However we will not assume unnecessary conditions in each of the propositions, lemmas, theorems and corollaries presented in the rest of this paper.

Assumption 1.3 There exists some finite θ such that $1 < \theta < r_A$ and $\theta = \delta(\boldsymbol{A}^*(\theta))$.

Remark 1.2 It is easy to see that $\delta(\boldsymbol{A}^*(1)) = 1$ and $\delta(\boldsymbol{A}^*(e^s))$ $(s < \log r_A)$ is the Perron-Frobenius eigenvalue of $\boldsymbol{A}^*(e^s)$. It is known [12] that $\delta(\boldsymbol{A}^*(e^s))$ is a non-decreasing convex function of s (see also Proposition 7 in [9]). Further it follows from Lemma 2.3.3 in [20] that under Assumption 1.1 (b), $\lim_{s\uparrow 0}(\mathrm{d}/\mathrm{d}s)\delta(\boldsymbol{A}^*(e^s)) = \rho$. Therefore Assumption 1.1 (b), (c) and Assumption 1.2 (a) are a necessary condition for Assumption 1.3, though they are not sufficient. In fact, $\lim_{y\uparrow r_A}\delta(\boldsymbol{A}^*(y))/y \le 1$ in some cases (such an example is given in Appendix E). A sufficient condition for Assumption 1.3 can be found in Theorem 4.12 in [6], and see also Section 3 in [9]. Finally, it should be noted that Assumption 1.3 requires

$$\frac{\mathrm{d}}{\mathrm{d}y}\delta(\boldsymbol{A}^*(y))\bigg|_{y=\theta} - 1 > 0, \tag{5}$$

which will be used to show that the prefactors of the asymptotic formulae presented in Section 3 are positive.

As is well known, the constant θ in Assumption 1.3 plays a role in the light-tailed asymptotic analysis of $\{\overline{\boldsymbol{x}}(k)\}$. Several researchers have studied the light-tailed asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ under the assumption of $\theta < r_B$, where $\{\overline{\boldsymbol{x}}(k)\}$ decays geometrically with rate $1/\theta$. Using the Tauberian theorem, Abate et al. [1] presented a necessary condition for

$$\lim_{k\to\infty}\theta^k\boldsymbol{x}(k) = \boldsymbol{d}, \tag{6}$$

where $\boldsymbol{d}$ is some positive vector. Note here that (6) yields

$$\lim_{k\to\infty}\theta^k\overline{\boldsymbol{x}}(k) = (\theta-1)^{-1}\boldsymbol{d}. \tag{7}$$

Møller [19] studied the asymptotic formula (6) by considering the inter-visit times of level zero, though his approach does not yield an explicit expression

of $\boldsymbol{d}$. Falkenberg [7] and Gail et al. [10] obtained the asymptotic formula (6) by locating the *dominant poles* (i.e., the maximum-order minimum-modulus poles; see Definition A.1.) of the generating function $\boldsymbol{x}^*(z)$ of $\{\boldsymbol{x}(k)\}$. However, Falkenberg's sufficient condition for (6) includes a redundant condition (see Remark 3.2).

Takine [26] presented geometric asymptotic formulae of the following form:

$$\lim_{n\to\infty} \theta^{nh+l}\overline{\boldsymbol{x}}(nh+l) = \boldsymbol{d}_l, \qquad l = 0, 1, \dots, h-1, \tag{8}$$

where h is some positive integer and $\boldsymbol{d}_l$'s ($l = 0, 1, \dots, h-1$) are some positive vectors. Clearly (8) includes (7) as a special case. For simplicity, the cases of $h = 1$ and $h \ge 2$ are called *the exactly geometric case* and *the periodically geometric case*, respectively. Using the Markov renewal approach, Takine [26] derived the expression of $\boldsymbol{d}$ for the exactly geometric case (i.e., $h = 1$) and that of $\boldsymbol{d}_l$ ($l = 0, 1, \dots, h-1$) for the periodically geometric case (i.e., $h \ge 2$), separately.

Li and Zhao [16] studied the light-tailed asymptotics of the stationary distribution of a Markov chain of GI/G/1 type. Corollary 1 therein would imply that the periodically geometric case is impossible, which is inconsistent with Lemma 3.1 in this paper. Therefore their results are valid only under the condition that excludes the periodically geometric case.

In all the studies mentioned above, it is assumed that the phase space is finite. Meanwhile, Miyazawa [17], Miyazawa and Zhao [18] and Li et al. [14] considered structured Markov chains with infinitely many phases, excluding the periodically geometric case. Miyazawa [17] and Miyazawa and Zhao [18] studied Markov chains of M/G/1 type and GI/G/1 type, respectively, and they derived asymptotic formulae like (6). Li et al. [14] derived an exactly geometric asymptotic formula for the stationary distribution of a quasi-birth-and-death (QBD) process and applied the obtained results to a generalized join-the-shortest-queue model. The infiniteness of the phase space causes some difficulties in asymptotic analysis of the stationary tail probability vectors. That is a challenging problem, but it is beyond the scope of this paper.

This paper studies the light-tailed asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ of a Markov chain of M/G/1 type, not excluding the periodically geometric case. The complex analysis approach used in this paper is basically the same as Falkenberg [7]'s approach, i.e., that is based on locating the dominant poles of $\overline{\boldsymbol{x}}^*(z)$. Indeed, the approach is classical, but it enables us to deal with the case of $\theta \ge r_B$ as well as the exactly and periodically geometric cases under the condition $\theta < r_B$ in a unified manner. In addition, the complex analysis approach gives us a deeper insight into the period in the light-tailed asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$, compared with the Markov renewal approach [17, 26]. In this paper, we first present a simple and unified formula for both the exactly geometric and periodically geometric cases,

assuming $\theta < r_B$. We also show that h in (8) is closely related to the period of a Markov additive process (MAdP) with kernel $\{\boldsymbol{A}(k+1); k = 0, \pm 1, \pm 2, \dots\}$, where $\boldsymbol{A}(k) = \boldsymbol{O}$ for $k \le -1$. As for the case of $\theta \ge r_B$, we derive light-tailed asymptotic formulae for $\{\overline{\boldsymbol{x}}(k)\}$ under some mild conditions. We can find no previous studies paying special attention to the case of $\theta \ge r_B$. As far as we know, only a few examples of this case have been shown in Li and Zhao [16]. Therefore this paper is the first comprehensive report of the light-tailed asymptotics in the case of $\theta \ge r_B$.

The rest of this paper is organized as follows. In section 2, we provide some preliminaries on $\{\overline{\boldsymbol{x}}(k)\}$ and the period of a MAdP related to the Markov chain of M/G/1 type. In section 3, we present light-tailed asymptotic formulae for three cases: $\theta < r_B$, $\theta > r_B$ and $\theta = r_B$ in subsections 3.1, 3.2 and 3.3, respectively. Further in the appendix, we describe fundamental results of the period of MAdPs, which play an important role in the asymptotic analysis of the stationary distributions of structured Markov chains such as ones of M/G/1 type and GI/G/1 type.

2 Preliminaries

Throughout this paper, we use the following conventions. Let $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$. Let $\mathbb{C}$ denote the set of complex numbers. Let ω denote a complex number such that $|\omega| = 1$. Let ι denote the imaginary unite, i.e., $\iota = \sqrt{-1}$. For any matrix $\boldsymbol{X}$ (resp. vector $\boldsymbol{y}$), its (i,j)th (resp. jth) element is denoted by $[\boldsymbol{X}]_{i,j}$ (resp. $[\boldsymbol{y}]_j$). When a (possibly complex-valued) function f and a nonnegative function g on $[0, \infty)$ satisfy $|f(x)| \le Cg(x)$ for any sufficiently large x, we write $f(x) = O(g(x))$. We also write $f(x) = o(g(x))$ if $\lim_{x \to \infty} |f(x)|/g(x) = 0$.

2.1 Some known results on the stationary tail probability vectors

Let $\boldsymbol{G}$ denote an $M \times M$ matrix whose (i,j)th ($i, j \in \mathbb{M}$) element represents $\Pr[S_{a(k)} = j \mid X_0 = k+1, S_0 = i]$ for any given k ($k \in \mathbb{N}$), where $a(k) = \inf\{n \in \mathbb{N}; X_n = k\}$. It is clear from the definition of $\boldsymbol{G}$ that

$$\boldsymbol{G} = \sum_{k=0}^{\infty} \boldsymbol{A}(k) \boldsymbol{G}^k. \tag{9}$$

It is known [20] that $\boldsymbol{G}$ is the minimal nonnegative solution of $\boldsymbol{X} = \sum_{k=0}^{\infty} \boldsymbol{A}(k) \boldsymbol{X}^k$. If Assumption 1.1 (b) and (c) hold, $\boldsymbol{G}$ is stochastic (see Theorem 2.3.1 in [20]).

The following result is an extension of Theorem 7.2.1 in [13] to the Markov chain of M/G/1 type.

Proposition 2.1 *If Assumption 1.1 (a) and (b) hold, then $\boldsymbol{G}$ is irreducible, or after some permutations it takes a form such that*

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{G}_1 & \boldsymbol{O} \\ \boldsymbol{G}_{\bullet,1} & \boldsymbol{G}_{\bullet} \end{pmatrix},$$

where $\boldsymbol{G}_1$ is irreducible and $\boldsymbol{G}_{\bullet}$ is strictly lower triangular.

Proof. See Appendix C.1. □

Remark 2.1 The proof of Proposition 2.1 does not require that $\boldsymbol{G}$ is stochastic. Thus Assumption 1.1 (c) is not needed.

Let $\boldsymbol{K}$ denote an $M_0 \times M_0$ matrix whose (i,j)th ($i,j \in \mathbb{M}_0$) element represents $\Pr[S_{a(0)} = j \mid X_0 = 0, S_0 = i]$. Matrix $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = \boldsymbol{B}(0) + \sum_{k=1}^{\infty} \boldsymbol{B}(k)\boldsymbol{G}^{k-1} \left(\boldsymbol{I} - \sum_{m=1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-1} \right)^{-1} \boldsymbol{C}(0),$$

and $\boldsymbol{x}(0)$ is given by

$$\boldsymbol{x}(0) = \left[1 + \frac{\boldsymbol{\kappa}}{1-\rho} \left\{ \boldsymbol{\beta}_B + \left(\boldsymbol{B} - \sum_{k=1}^{\infty} \boldsymbol{B}(k)\boldsymbol{G}^{k} \right) [\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{e}\boldsymbol{\pi}]^{-1} \boldsymbol{\beta}_A \right\} \right]^{-1} \boldsymbol{\kappa},$$

where $\boldsymbol{\kappa}$ denotes the stationary probability vector of $\boldsymbol{K}$ (see Theorem 3.1 in [22]). Further $\boldsymbol{x}(k)$ ($k = 1, 2, \dots$) is determined by Ramaswami's [21] recursion:

$$\boldsymbol{x}(k) = \left(\boldsymbol{x}(0)\boldsymbol{U}_0(k) + \sum_{j=1}^{k-1} \boldsymbol{x}(j)\boldsymbol{U}(k-j) \right) (\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \tag{10}$$

where $\boldsymbol{U}(k)$ ($k = 0, 1, \dots$) and $\boldsymbol{U}_0(k)$ ($k = 1, 2, \dots$) are given by

$$\boldsymbol{U}(k) = \sum_{m=k+1}^{\infty} \boldsymbol{A}(m)\boldsymbol{G}^{m-k-1}, \qquad \boldsymbol{U}_0(k) = \sum_{m=k}^{\infty} \boldsymbol{B}(m)\boldsymbol{G}^{m-k}, \tag{11}$$

respectively. Note here that for any fixed $k \in \mathbb{N}$,

$$[\boldsymbol{U}(0)]_{i,j} = \Pr[S_{a(k)} = j, X_n \ge k \ (n = 1, 2, \dots, a(k)) \mid X_0 = k, S_0 = i], \qquad i,j \in \mathbb{M},$$

and thus $\boldsymbol{I} - \boldsymbol{U}(0)$ is nonsingular due to Assumption 1.1 (a).

We now define $\boldsymbol{R}(k)$ and $\boldsymbol{R}_0(k)$ ($k \in \mathbb{N}$) as

$$\boldsymbol{R}(k) = \boldsymbol{U}(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \qquad \boldsymbol{R}_0(k) = \boldsymbol{U}_0(k)(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}, \tag{12}$$

respectively. For convenience, let $\boldsymbol{R}(0) = \boldsymbol{O}$. We then rewrite (10) as

$$\boldsymbol{x}(k) = \boldsymbol{x}(0)\boldsymbol{R}_0(k) + \sum_{j=1}^{k} \boldsymbol{x}(j)\boldsymbol{R}(k-j). \tag{13}$$

Let $\boldsymbol{R}^*(z)$ and $\boldsymbol{R}_0^*(z)$ denote the generating functions of $\{\boldsymbol{R}(k)\}$ and $\{\boldsymbol{R}_0(k)\}$ defined by $\sum_{k=0}^{\infty} z^k \boldsymbol{R}(k)$ and $\sum_{k=1}^{\infty} z^k \boldsymbol{R}_0(k)$, respectively. It then follows from (13) that

$$\boldsymbol{x}^*(z) = \boldsymbol{x}(0)\boldsymbol{R}_0^*(z)[\boldsymbol{I} - \boldsymbol{R}^*(z)]^{-1},$$

from which and (3) we have

$$\overline{\boldsymbol{x}}^*(z) = \frac{\boldsymbol{x}^*(1) - \boldsymbol{x}(0)\boldsymbol{R}_0^*(z)[\boldsymbol{I} - \boldsymbol{R}^*(z)]^{-1}}{1-z}.$$

Using (9), (11) and (12), we can readily have the following result.

Proposition 2.2 *If Assumption 1.1 (a) and (b) hold, then*

$$\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z) = (\boldsymbol{I} - \boldsymbol{R}^*(z))(\boldsymbol{I} - \boldsymbol{U}(0))(\boldsymbol{I} - \boldsymbol{G}/z), \quad 0 < |z| < r_A, \tag{14}$$

$$\boldsymbol{B}^*(z) - \boldsymbol{U}_0(1)\boldsymbol{G} = \boldsymbol{R}_0^*(z)(\boldsymbol{I} - \boldsymbol{U}(0))(\boldsymbol{I} - \boldsymbol{G}/z), \qquad 0 < |z| < r_B, \tag{15}$$

where

$$\boldsymbol{\Gamma}_A^*(z) = z^{-1}\boldsymbol{A}^*(z). \tag{16}$$

Remark 2.2 Equation (14) shows the RG-factorization of the Markov chain of M/G/1 type (see, Theorem 14 in [28]).

Proposition 2.3 *Suppose Assumption 1.1 (a), (b) and Assumption 1.2 (a) hold. Then*

$$\det(\boldsymbol{I}-\boldsymbol{\Gamma}_A^*(z)) = 0 \textit{ if and only if } \det(\boldsymbol{I}-\boldsymbol{R}^*(z)) = 0, \qquad 1 < |z| < r_A. \tag{17}$$

Proof. Since $\boldsymbol{I} - \boldsymbol{U}(0)$ and $\boldsymbol{I} - \boldsymbol{G}/z$ are nonsingular for $|z| > 1$, (14) leads to (17). □

It follows from (16) that $\boldsymbol{\Gamma}_A^*(\theta) = \boldsymbol{A}^*(\theta)/\theta$ and thus Assumption 1.3 yields

$$\delta(\boldsymbol{\Gamma}_A^*(\theta)) = \delta(\boldsymbol{A}^*(\theta))/\theta = 1. \tag{18}$$

The matrix $\boldsymbol{\Gamma}_A^*(\theta)$ is irreducible, because $[\boldsymbol{A}^*(\theta)]_{i,j} > 0$ if and only if $[\boldsymbol{A}^*(1)]_{i,j} = [\boldsymbol{A}]_{i,j} > 0$. Therefore $\delta(\boldsymbol{\Gamma}_A^*(\theta)) = 1$ is the Perron-Frobenius eigenvalue of $\boldsymbol{\Gamma}_A^*(\theta)$. Let $\boldsymbol{\mu}(\theta)$ and $\boldsymbol{v}(\theta)$ denote the Perron-Frobenius left- and right-eigenvectors of $\boldsymbol{\Gamma}_A^*(\theta)$, which are normalized such that

$$\boldsymbol{\mu}(\theta)\boldsymbol{e} = 1, \qquad \boldsymbol{\mu}(\theta)\boldsymbol{v}(\theta) = 1.$$

Clearly $\boldsymbol{\mu}(\theta) > \mathbf{0}$ and $\boldsymbol{v}(\theta) > \mathbf{0}$. Further (14) yields $\boldsymbol{\mu}(\theta)\boldsymbol{R}^*(\theta) = \boldsymbol{\mu}(\theta)$. It thus follows from Corollary 8.1.30 in [11] that

$$\delta(\boldsymbol{R}^*(\theta)) = 1. \tag{19}$$

Using this, we can prove the following result.

Proposition 2.4 *If Assumption 1.1 (a)–(c), Assumption 1.2 (a) and Assumption 1.3 hold, then $\boldsymbol{R} \triangleq \boldsymbol{R}^*(1)$ is irreducible, or after some permutations it takes a form such that*

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_{1,\bullet} \\ \boldsymbol{O} & \boldsymbol{R}_\bullet \end{pmatrix},$$

where $\boldsymbol{R}_1$ is irreducible and $\boldsymbol{R}_\bullet$ is strictly upper triangular.

Proof. This proposition can be proved in a similar way to the proof of Theorem 7.2.2 in [13], but some modifications are needed. A complete proof is given in Appendix C.2. □

Remark 2.3 Because of some dual properties between $\boldsymbol{G}$ and $\boldsymbol{R}$, it might seem that the statement of Proposition 2.4 would be true under the conditions of Proposition 2.1, i.e., Assumption 1.1 (a) and (b). In fact, if $\boldsymbol{T}$ is the transition probability matrix of a QBD, the statement of Proposition 2.4 is true under Assumption 1.1 (a) and (b) (see Theorems 7.2.1 and 7.2.2 in [13]). However, this is not the case in general. Assume that $\boldsymbol{A}(k) = \boldsymbol{O}$ for all $k = 2, 3, \dots$ and the other block matrices of $\boldsymbol{T}$ in (1) are all positive. Then Assumption 1.1 (a) and (b) are satisfied, whereas $\boldsymbol{R} = \boldsymbol{O}$.

Since $[\boldsymbol{R}]_{i,j} > 0$ implies $[\boldsymbol{R}^*(\theta)]_{i,j} > 0$ (vice versa), it follows from (19) and Proposition 2.4 that $\delta(\boldsymbol{R}^*(\theta)) = 1$ is a simple eigenvalue. Let $\boldsymbol{s}(\theta) = (\boldsymbol{I} - \boldsymbol{U}(0))(\boldsymbol{I} - \boldsymbol{G}/\theta)\boldsymbol{v}(\theta)$. From (14), we then have $\boldsymbol{R}^*(\theta)\boldsymbol{s}(\theta) = \boldsymbol{s}(\theta)$. Note here that

$$(\boldsymbol{I} - \boldsymbol{G}/\theta)^{-1}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1}\boldsymbol{s}(\theta) = \boldsymbol{v}(\theta) > \mathbf{0},$$
$$(\boldsymbol{I} - \boldsymbol{G}/\theta)^{-1}(\boldsymbol{I} - \boldsymbol{U}(0))^{-1} \geq \boldsymbol{O} \neq \boldsymbol{O}.$$

Thus $\boldsymbol{s}(\theta) \geq \mathbf{0}, \neq \mathbf{0}$ (see, e.g., Theorem 8.3.1 in [11]). The following proposition summarizes the results on the spectral radius of $\boldsymbol{R}^*(\theta)$ and its corresponding eigenvectors.

Proposition 2.5 (Lemma 5 in [16]) *Suppose Assumption 1.1 (a)–(c), Assumption 1.2 (a) and Assumption 1.3 hold. Then $\boldsymbol{\mu}(\theta) > \mathbf{0}$ and $\boldsymbol{s}(\theta) \ge \mathbf{0}, \neq \mathbf{0}$, which are left- and right-eigenvectors, respectively, of $\boldsymbol{R}^*(\theta)$ corresponding simple eigenvalue $\delta(\boldsymbol{R}^*(\theta)) = 1$.*

We define $\boldsymbol{\Gamma}_R(k)$ ($k \in \mathbb{Z}$) as

$$\boldsymbol{\Gamma}_R(k) = \begin{cases} \theta^k \mathrm{diag}(\boldsymbol{\mu}(\theta))^{-1} \boldsymbol{R}(k)^{\mathrm{t}} \mathrm{diag}(\boldsymbol{\mu}(\theta)), & k = 1, 2, \ldots, \\ \boldsymbol{O}, & k = 0, -1, -2, \ldots, \end{cases} \tag{20}$$

where super-subscript t denotes transpose and $\mathrm{diag}(\boldsymbol{\mu}(\theta))$ denotes an $M \times M$ diagonal matrix whose jth diagonal element is equal to $[\boldsymbol{\mu}(\theta)]_j$. Let $\boldsymbol{\Gamma}^*_R(z)$ denote the generating function of $\{\boldsymbol{\Gamma}_R(k)\}$ defined by $\sum_{k \in \mathbb{Z}} z^k \boldsymbol{\Gamma}_R(k)$. We then have

$$\boldsymbol{\Gamma}^*_R(z) = \mathrm{diag}(\boldsymbol{\mu}(\theta))^{-1} \boldsymbol{R}^*(\theta z)^{\mathrm{t}} \mathrm{diag}(\boldsymbol{\mu}(\theta)). \tag{21}$$

It is easy to see that $\boldsymbol{\Gamma}^*_R(1)$ is stochastic. Further it follows from Propositions 2.4 and 2.5 that either $\boldsymbol{\Gamma}^*_R(1)$ is irreducible or after some permutations, $\boldsymbol{\Gamma}^*_R(z)$ takes a form such that

$$\boldsymbol{\Gamma}^*_R(z) = \begin{array}{c} \\ \mathbb{M}^{(1)} \\ \mathbb{M}^{(2)} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad\mathbb{M}^{(1)} & \quad\quad\mathbb{M}^{(2)} \end{array} \\ \begin{pmatrix} \boldsymbol{\Gamma}^*_{R,1}(z) & \boldsymbol{O} \\ \boldsymbol{\Gamma}^*_{R,2,1}(z) & \boldsymbol{\Gamma}^*_{R,2}(z) \end{pmatrix} \end{array},$$

where $\boldsymbol{\Gamma}^*_{R,1}(1)$ is irreducible and stochastic and $\boldsymbol{\Gamma}^*_{R,2}(z)$ is strictly lower triangular, i.e., $\delta(\boldsymbol{\Gamma}^*_{R,2}(z)) = 0$. Thus according to Theorem B.1, we define h as

$$h = \max\{n \in \mathbb{N}; \delta(\boldsymbol{\Gamma}^*_R(\omega_n)) = 1\}, \tag{22}$$

where $\omega_x = \exp(2\pi\iota/x)$ for $x \ge 1$. Note that h is the period of the recurrent class $\mathbb{M}^{(1)}$ in an MAdP $\{\boldsymbol{\Gamma}_R(k); k \in \mathbb{Z}\}$. Let $h_{i,j}$ ($i, j \in \mathbb{M}$) denote the first jump point of the distribution of the first passage time from state i to state j in the MAdP $\{\boldsymbol{\Gamma}_R(k); k \in \mathbb{Z}\}$. Then Takine [26]'s asymptotic formulae are as follows.

Proposition 2.6 (Theorem 2 in [26]) *Suppose Assumptions 1.1–1.3 hold and $\theta < r_B$. Let $\boldsymbol{\pi}_* = \sum_{k=1}^{\infty} \boldsymbol{x}(k)$, which is given by (see Lemma 3 in [26])*

$$\boldsymbol{\pi}_* = [\boldsymbol{x}(0)\{\boldsymbol{B} + \boldsymbol{\beta}_B \boldsymbol{g}\} - \boldsymbol{x}(1)\boldsymbol{A}(0)](\boldsymbol{I} - \boldsymbol{A} + (\boldsymbol{e} - \boldsymbol{\beta}_A)\boldsymbol{g})^{-1},$$

where $\boldsymbol{g}$ denotes the stationary probability vector of $\boldsymbol{G}$. If $h \ge 2$, then for $l = 0, 1, \ldots, h-1$,

$$\lim_{n \to \infty} \theta^{nh+l} [\overline{\boldsymbol{x}}(nh + l)]_j = [\boldsymbol{d}_l]_j, \qquad j \in \mathbb{M},$$

where

$$[\boldsymbol{d}_l]_j = \frac{h}{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta} - 1} \sum_{\nu\in\mathbb{M}} \sum_{k=0}^{\infty} \theta^{l+kh-h_{j,\nu}}$$
$$\cdot \sum_{m=l+kh-h_{j,\nu}+1}^{\infty} [\boldsymbol{x}(0)\boldsymbol{R}_0(m) + \boldsymbol{\pi}_*\boldsymbol{R}(m)]_\nu [(\boldsymbol{I} - \boldsymbol{U}(0))(\boldsymbol{I} - \boldsymbol{G}/\theta)\boldsymbol{v}(\theta)]_\nu [\boldsymbol{\mu}(\theta)]_j.$$

Proposition 2.7 (Theorem 3 in [26]) *Suppose Assumptions 1.1–1.3 hold and $\theta < r_B$. If $h = 1$, then*

$$\lim_{k\to\infty} \theta^k \overline{\boldsymbol{x}}(k) = \frac{[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta) - \boldsymbol{x}(1)\boldsymbol{A}(0)]\,\boldsymbol{v}(\theta)}{(\theta-1)\{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta} - 1\}} \cdot \boldsymbol{\mu}(\theta). \tag{23}$$

2.2 Period of a related Markov additive process

We consider a MAdP $\{(\breve{X}_n, \breve{S}_n); n = 0, 1, \dots\}$ with state space $\mathbb{Z} \times \mathbb{M}$ and kernel $\{\boldsymbol{\Gamma}_A(k); k \in \mathbb{Z}\}$, where

$$\boldsymbol{\Gamma}_A(k) = \begin{cases} \boldsymbol{A}(k+1), & k = -1, 0, 1, \dots, \\ \boldsymbol{O}, & k = -2, -3, -4, \dots. \end{cases} \tag{24}$$

It follows from (16) and (24) that $\sum_{k\in\mathbb{Z}} z^k \boldsymbol{\Gamma}_A(k) = \boldsymbol{\Gamma}_A^*(z)$. It is easy to see that for $i, j \in \mathbb{M}$,

$$\begin{aligned} &\Pr[\breve{X}_{n+1} = k_* + k, \breve{S}_{n+1} = j \mid \breve{X}_n = k_*, \breve{S}_n = i] \\ &\quad = \Pr[X_{n+1} = k_* + k, S_{n+1} = j \mid X_n = k_*, S_n = i], \quad k_* + k \in \mathbb{N},\ k_* \in \mathbb{N}. \end{aligned} \tag{25}$$

For any two states (k_1, j_1) and (k_2, j_2) in $\mathbb{Z} \times \mathbb{M}$, we write $(k_1, j_1) \to (k_2, j_2)$ when there exists a path from (k_1, j_1) to (k_2, j_2) with some positive probability.

Proposition 2.8 *Suppose Assumption 1.1 (a) and (b) hold. Then for each $j \in \mathbb{M}$, there exists a nonzero integer k_j such that $(0, j) \to (k_j, j)$.*

Proof. See Appendix C.3. □

Assumption 1.1 (b) and (16) show that $\boldsymbol{\Gamma}_A^*(1) = \boldsymbol{A}$ is irreducible, from which and Proposition 2.8 it follows that the period of the MAdP $\{(\breve{X}_n, \breve{S}_n); n = 0, 1, \dots\}$ is well-defined and is denoted by τ (see Definition B.1). Note here that from (25) and the definition of $\boldsymbol{G}$,

$$[\boldsymbol{G}]_{i,j} = \Pr[\breve{S}_{\breve{a}(k)} = j \mid \breve{X}_0 = k+1, \breve{S}_0 = i], \qquad i, j \in \mathbb{M},$$

where $\breve{a}(k) = \inf\{n \in \mathbb{N}; \breve{X}_n = k\}$. Note also that $\boldsymbol{G}$ has no zero rows due to Assumption 1.1 (a). Proposition 2.1 then implies that there exists a path such that

$$(k, j_0) \to (k-1, j_1) \to (k-2, j_2) \to \cdots \to (k-M, j_M),$$

where $k \in \mathbb{Z}$ and $j_n \in \mathbb{M}$ for $n = 0, 1, \dots, M$. In the above path, a phase appears at least two times, and thus $\tau \le M$.

Proposition 2.9 (Propositions 13 and 14 in [8]) *Under Assumption 1.1 (a) and (b),*

$$\tau = \max\{n \in \mathbb{M}; \delta(\boldsymbol{\Gamma}_A^*(\omega_n)) = 1\}.$$

Lemma B.2 shows that there exists a function[‡1] p from $\mathbb{M}$ to $\{0, 1, \dots, \tau - 1\}$ such that

$$[\boldsymbol{\Gamma}_A(k)]_{i,j} > 0 \text{ only if } k \equiv p(j) - p(i) \pmod{\tau}.$$

Let $\boldsymbol{\Delta}_M(z)$ denote an $M \times M$ diagonal matrix such that

$$\boldsymbol{\Delta}_M(z) = \begin{pmatrix} z^{-p(1)} & & & \\ & z^{-p(2)} & & \\ & & \ddots & \\ & & & z^{-p(M)} \end{pmatrix}.$$

Let $\boldsymbol{\mu}(z)$ and $\boldsymbol{v}(z)$ ($z \in \mathbb{C}$) denote left- and right-eigenvectors of $\boldsymbol{\Gamma}_A^*(z)$ corresponding to eigenvalue $\delta(\boldsymbol{\Gamma}_A^*(z))$, normalized such that

$$\boldsymbol{\mu}(z)\boldsymbol{\Delta}_M(z/|z|)\boldsymbol{e} = 1, \qquad \boldsymbol{\mu}(z)\boldsymbol{v}(z) = 1.$$

Note that if z is real, $\boldsymbol{\Delta}_M(z/|z|) = \boldsymbol{I}$ and therefore the definition of $\boldsymbol{\mu}(z)$ and $\boldsymbol{v}(z)$ is consistent with that of $\boldsymbol{\mu}(\theta)$ and $\boldsymbol{v}(\theta)$.

The following proposition is an immediate consequence of Lemma B.3.

Proposition 2.10 *If Assumption 1.1 (a) and (b) hold, then for* $0 < y < r_A$*,*

$$\delta(\boldsymbol{\Gamma}_A^*(y\omega_\tau^\nu)) = \delta(\boldsymbol{\Gamma}_A^*(y)), \qquad \nu = 0, 1, \dots, \tau - 1, \tag{26}$$

$$\boldsymbol{\mu}(y\omega_\tau^\nu) = \boldsymbol{\mu}(y)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}, \qquad \nu = 0, 1, \dots, \tau - 1, \tag{27}$$

$$\boldsymbol{v}(y\omega_\tau^\nu) = \boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(y), \qquad \nu = 0, 1, \dots, \tau - 1. \tag{28}$$

It follows from (16) that

$$\delta(\boldsymbol{\Gamma}_A^*(y)) = y^{-1}\delta(\boldsymbol{A}^*(y)), \qquad 0 < y < r_A. \tag{29}$$

Thus according to the property of $\delta(\boldsymbol{A}^*(y))$ (see Remark 1.2), we obtain

$$\delta(\boldsymbol{\Gamma}_A^*(y)) < 1, \qquad 1 < y < \theta. \tag{30}$$

Further from (29), we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\delta(\boldsymbol{\Gamma}_A^*(z))\bigg|_{z=\theta} = \frac{\mathrm{d}}{\mathrm{d}y}\delta(\boldsymbol{\Gamma}_A^*(y))\bigg|_{y=\theta} = \frac{1}{\theta}\left(\frac{\mathrm{d}}{\mathrm{d}y}\delta(\boldsymbol{A}^*(y))\bigg|_{y=\theta} - 1\right) > 0, \tag{31}$$

[‡1] corrected: “an injective function” $\longrightarrow$ “a function”

where the second equality follows from (18) and the last inequality follows from (5).

The following proposition can be easily obtained by (18), Proposition 2.9 and Theorem B.1.

Proposition 2.11 *Suppose Assumption 1.1 (a)–(c), Assumption 1.2 (a) and Assumption 1.3 hold. Then* $\delta(\boldsymbol{\Gamma}_A^*(\theta\omega)) = 1$ *if and only if* $\omega^\tau = 1$. *Thus*

$$\tau = \max\{n \in \mathbb{M}; \delta(\boldsymbol{\Gamma}_A^*(\theta\omega_n)) = 1\}.$$

Further if $\delta(\boldsymbol{\Gamma}_A^*(\theta\omega)) = 1$*, the eigenvalue is simple.*

Proposition 2.12 *If Assumption 1.1 (a)–(c), Assumption 1.2 (a) and Assumption 1.3 hold, then* τ *is equal to period* h *of the MAdP with kernel* $\{\boldsymbol{\Gamma}_{R,1}(k); k \in \mathbb{Z}\}$.

Proof. It follows from (22) and Proposition 2.11 that $h = \tau$ if the following is true.

$$\delta(\boldsymbol{\Gamma}_R^*(\omega)) = 1 \text{ if and only if } \delta(\boldsymbol{\Gamma}_A^*(\theta\omega)) = 1. \tag{32}$$

In fact, (21) shows that

$$\delta(\boldsymbol{\Gamma}_R^*(\omega)) = 1 \text{ if and only if } \delta(\boldsymbol{R}^*(\theta\omega)) = 1. \tag{33}$$

Since $\delta(\boldsymbol{\Gamma}_A^*(\theta)) = \delta(\boldsymbol{R}^*(\theta)) = 1$, $|\delta(\boldsymbol{\Gamma}_A^*(\theta\omega))| \le 1$ and $|\delta(\boldsymbol{R}^*(\theta\omega))| \le 1$ (see Theorem 8.1.18 in [11]). Therefore Proposition 2.3 implies that

$$\delta(\boldsymbol{R}^*(\theta\omega)) = 1 \text{ if and only if } \delta(\boldsymbol{\Gamma}_A^*(\theta\omega)) = 1.$$

This and (33) yield (32). □

3 Main Results

Proposition 1.1 and Definition 1.1 show that $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$) is holomorphic in the domain $\{z \in \mathbb{C}; |z| \le 1\}$ and thus has no pole in the same domain. It follows from (3), (4) and (16) that for z such that $[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)]^{-1}$ exists,

$$\overline{\boldsymbol{x}}^*(z) = \frac{\boldsymbol{x}^*(1)}{1-z} - \frac{\boldsymbol{x}(0)\boldsymbol{B}^*(z) - \boldsymbol{x}(1)\boldsymbol{A}(0)}{1-z}\left[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)\right]^{-1}. \tag{34}$$

By definition,

$$[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)]^{-1} = \frac{\mathrm{adj}(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z))}{\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z))}, \tag{35}$$

where $\mathrm{adj}(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z))$ denotes the adjugate of $\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)$. Therefore the roots of $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = 0$ with $|z| > 1$, if any, are candidates for the dominant poles of $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$).

Let $r_m(z)$'s ($m = 1, 2, \dots, M$) denote the eigenvalues of $\boldsymbol{\Gamma}_A^*(z)$ such that

$$r_1(z) = \delta(\boldsymbol{\Gamma}_A^*(z)), \qquad |r_1(z)| \ge |r_2(z)| \ge \cdots \ge |r_M(z)|, \tag{36}$$

$$\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = \prod_{m=1}^{M} (1 - r_m(z)). \tag{37}$$

For convenience in what follows, let ε_0 denote a sufficiently small positive number, which may take different values in different places.

Lemma 3.1 *Suppose Assumption 1.1 (a)–(c), Assumption 1.2 (a) and Assumption 1.3 hold. Then the equation* $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = 0$ *has exactly* τ *roots* $\theta\omega_\tau^\nu$ *(*$\nu = 0, 1, \dots, \tau - 1$*) in the domain* $\{z \in \mathbb{C}; 1 < |z| < \theta + \varepsilon_0\}$*. In addition, each of the roots is simple.*

Proof. It follows from (26) and (31) that $(\mathrm{d}/\mathrm{d}z)\delta(\boldsymbol{\Gamma}_A^*(z))|_{z=\theta\omega_\tau^\nu} > 0$ for $\nu = 0, 1, \dots, \tau - 1$. Thus, according to the former part of Proposition 2.11, $\{\theta\omega_\tau^\nu; \nu = 0, 1, \dots, \tau - 1\}$ are simple roots of the equation $\delta(\boldsymbol{\Gamma}_A^*(z)) - 1 = 0$. Further the latter part of the proposition implies that $\prod_{m=2}^{M}(1 - r_m(\theta\omega_\tau^\nu)) \neq 0$ for $\nu = 0, 1, \dots, \tau - 1$. Therefore $\{\theta\omega_\tau^\nu; \nu = 0, 1, \dots, \tau - 1\}$ are simple roots of the equation $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = 0$. Note here that $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z))$ is holomorphic and not identically zero in the domain $\{z \in \mathbb{C}; 0 < |z| < r_A\}$. As a result, it suffices to show that $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = 0$ has no other roots in the domain $\{z \in \mathbb{C}; 1 < |z| \le \theta\}$.

By definition, if $\delta(\boldsymbol{\Gamma}_A^*(\theta\omega)) \neq 1$, then $r_m(\theta\omega) \neq 1$ for all $m = 2, 3, \dots, M$. Thus $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(\theta\omega)) = 0$ only if $\omega = \omega_\tau^\nu$ ($\nu = 0, 1, \dots, \tau - 1$). In addition, it follows from (30) and Theorem 8.1.18 in [11] that for $m = 2, 3, \dots, M$,

$$|r_m(z)| \le |\delta(\boldsymbol{\Gamma}_A^*(z))| \le \delta(\boldsymbol{\Gamma}_A^*(|z|)) < 1, \qquad 1 < |z| < \theta.$$

This and (37) show that $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) \neq 0$ in the domain $\{z \in \mathbb{C}; 1 < |z| < \theta\}$. □

Remark 3.1 In the proof of Corollary 1 in [16], it is stated that $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = 0$ has one and only one root on the circle $\{z \in \mathbb{C}; |z| = \theta\}$, which is, in general, incorrect.

Lemma 3.2 *Suppose Assumptions 1.1 (a)–(c), Assumption 1.2 (a) and Assumption 1.3 hold. Then for* $\nu = 0, 1, \dots, \tau - 1$*,*

$$\mathrm{adj}(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(\theta\omega_\tau^\nu)) = \prod_{m=2}^{M} (1 - r_m(\theta\omega_\tau^\nu)) \boldsymbol{v}(\theta\omega_\tau^\nu) \boldsymbol{\mu}(\theta\omega_\tau^\nu) \neq \boldsymbol{O},$$
$$\nu = 0, 1, \dots, \tau - 1. \tag{38}$$

Proof. See Appendix C.4. □

In the rest of this section, we first derive a light-tailed asymptotic formula for the case of $\theta < r_B$ in subsection 3.1. We then discuss the cases of $\theta > r_B$ and $\theta = r_B$ in subsections 3.2 and 3.3, respectively.

3.1 Case of $\theta < r_B$

Lemmas 3.1 and 3.2 imply that if $\theta < \min(r_A, r_B)$, $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$) is holomorphic for $|z| < \theta$ and meromorphic for $|z| \le \theta$, and thus the candidates for the dominant poles of $[\overline{\boldsymbol{x}}^*(z)]_j$ are the simple roots $\{\theta\omega_\tau^\nu; \nu = 0, 1, \dots, \tau-1\}$ of $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A(z)) = 0$. Note here that $z = \theta\omega_\tau^\nu$ ($\nu = 0, 1, \dots, \tau - 1$) is a simple pole of $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$) if and only if

$$0 < \lim_{z \to \theta\omega_\tau^\nu} \left| \left(1 - \frac{z}{\theta\omega_\tau^\nu}\right) [\overline{\boldsymbol{x}}^*(z)]_j \right| < \infty.$$

Thus it follows from Theorem A.1 and Remark A.3 that if each $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$) has at least one pole of $\{\theta\omega_\tau^\nu; \nu = 0, 1, \dots, \tau - 1\}$, then

$$\overline{\boldsymbol{x}}(k) = \theta^{-k} \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^k} \lim_{z \to \theta\omega_\tau^\nu} \left(1 - \frac{z}{\theta\omega_\tau^\nu}\right) \overline{\boldsymbol{x}}^*(z) + O((\theta + \varepsilon_0)^{-k}) \boldsymbol{e}^{\mathrm{t}}, \qquad (39)$$

where the dominant term on the right hand side of (39) is positive for any $k = 0, 1, \dots$ (see Theorem A.1 (d)), i.e., for any $j \in \mathbb{M}$ and $k = 0, 1, \dots,$

$$\left[\theta^{-k} \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^k} \lim_{z \to \theta\omega_\tau^\nu} \left(1 - \frac{z}{\theta\omega_\tau^\nu}\right) \overline{\boldsymbol{x}}^*(z)\right]_j > 0. \qquad (40)$$

We now note that (34) yields

$$\lim_{z \to \theta\omega_\tau^\nu} \left(1 - \frac{z}{\theta\omega_\tau^\nu}\right) \overline{\boldsymbol{x}}^*(z) = \frac{\boldsymbol{x}(0)\boldsymbol{B}^*(\theta\omega_\tau^\nu) - \boldsymbol{x}(1)\boldsymbol{A}(0)}{\theta\omega_\tau^\nu - 1} \cdot \lim_{z \to \theta\omega_\tau^\nu} \left(1 - \frac{z}{\theta\omega_\tau^\nu}\right) [\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)]^{-1}. \qquad (41)$$

We then obtain the following result.

Lemma 3.3 *Suppose Assumptions 1.1 (a)–(c), Assumption 1.2 (a) and Assumption 1.3 hold. Then for* $\nu = 0, 1, \dots, \tau - 1$,

$$\lim_{z \to \theta\omega_\tau^\nu} \left(1 - \frac{z}{\theta\omega_\tau^\nu}\right) [\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)]^{-1} = \frac{\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta)\boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}}{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta} - 1}. \qquad (42)$$

Proof. It follows from (35), (37) and (38) that

$$\lim_{z\to\theta\omega_\tau^\nu}\left(1-\frac{z}{\theta\omega_\tau^\nu}\right)[\boldsymbol{I}-\boldsymbol{\Gamma}_A^*(z)]^{-1} = \lim_{z\to\theta\omega_\tau^\nu}\frac{1-\dfrac{z}{\theta\omega_\tau^\nu}}{1-r_1(z)}\cdot\lim_{z\to\theta\omega_\tau^\nu}\frac{\mathrm{adj}(\boldsymbol{I}-\boldsymbol{\Gamma}_A^*(z))}{\prod_{m=2}^{M}(1-r_m(z))}$$
$$= \frac{1}{\theta\omega_\tau^\nu}\frac{1}{(\mathrm{d}/\mathrm{d}z)r_1(z)|_{z=\theta\omega_\tau^\nu}}\cdot\boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu), \tag{43}$$

where we use l'Hôpital's rule in the second equality. Note that (26), (31) and (37) yield

$$\theta\omega_\tau^\nu\frac{\mathrm{d}}{\mathrm{d}z}r_1(z)\bigg|_{z=\theta\omega_\tau^\nu} = \theta\frac{\mathrm{d}}{\mathrm{d}z}r_1(z)\bigg|_{z=\theta} = \frac{\mathrm{d}}{\mathrm{d}y}\delta(\boldsymbol{A}^*(y))\bigg|_{y=\theta} - 1,\quad \nu=0,1,\dots,\tau-1.$$

Thus (43) leads to

$$\lim_{z\to\theta\omega_\tau^\nu}\left(1-\frac{z}{\theta\omega_\tau^\nu}\right)[\boldsymbol{I}-\boldsymbol{\Gamma}_A^*(z)]^{-1} = \frac{\boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu)}{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta}-1},\quad \nu=0,1,\dots,\tau-1.$$

Finally, substituting (27) and (28) into the above equation, we obtain (42). □

Applying Lemma 3.3 to (41), we have

$$\lim_{z\to\theta\omega_\tau^\nu}\left(1-\frac{z}{\theta\omega_\tau^\nu}\right)\overline{\boldsymbol{x}}^*(z) = c(\omega_\tau^\nu)\cdot\boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1},\qquad \nu=0,1,\dots,\tau-1, \tag{44}$$

where $c(\omega_\tau^\nu)$ ($\nu=0,1,\dots,\tau-1$) is a scalar such that

$$c(\omega_\tau^\nu) = \frac{[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta\omega_\tau^\nu)-\boldsymbol{x}(1)\boldsymbol{A}(0)]\,\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta)}{(\theta\omega_\tau^\nu-1)\{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta}-1\}}. \tag{45}$$

It follows from (44) that $c(\omega_\tau^\nu)\neq 0$ if and only if $z=\theta\omega_\tau^\nu$ is a simple pole of $[\overline{\boldsymbol{x}}^*(z)]_j$ for any $j\in\mathbb{M}$.

Lemma 3.4 *Suppose Assumptions 1.1–1.3 hold and $\theta < r_B$. Then $c(\omega_\tau^0) = c(1) > 0$ and therefore $z=\theta$ is a simple pole of $[\overline{\boldsymbol{x}}^*(z)]_j$ for any $j\in\mathbb{M}$.*

Proof. From (45), we have

$$c(1) = \frac{1}{(\theta-1)\{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta}-1\}}\cdot[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta)-\boldsymbol{x}(1)\boldsymbol{A}(0)]\boldsymbol{v}(\theta).$$

Thus according to (5) and $\theta-1>0$, it suffices to show that

$$[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta)-\boldsymbol{x}(1)\boldsymbol{A}(0)]\boldsymbol{v}(\theta) > 0. \tag{46}$$

From (10) and $\boldsymbol{G}=(\boldsymbol{I}-\boldsymbol{U}(0))^{-1}\boldsymbol{A}(0)$, we have $\boldsymbol{x}(1)\boldsymbol{A}(0)=\boldsymbol{x}(0)\boldsymbol{U}_0(1)\boldsymbol{G}$ and therefore

$$[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta)-\boldsymbol{x}(1)\boldsymbol{A}(0)]\boldsymbol{v}(\theta) = \boldsymbol{x}(0)\left(\boldsymbol{B}^*(\theta)-\boldsymbol{U}_0(1)\boldsymbol{G}\right)\boldsymbol{v}(\theta). \tag{47}$$

Letting $z = \theta$ in (15) and substituting it into the right hand side of (47), we have

$$[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta) - \boldsymbol{x}(1)\boldsymbol{A}(0)]\boldsymbol{v}(\theta) = \boldsymbol{x}(0)\boldsymbol{R}_0^*(\theta)\boldsymbol{s}(\theta),$$

where $\boldsymbol{s}(\theta) = (\boldsymbol{I} - \boldsymbol{U}(0))(\boldsymbol{I} - \boldsymbol{G}/\theta)\boldsymbol{v}(\theta) \ge \mathbf{0}, \ne \mathbf{0}$ (see Proposition 2.5). Note here that $[\boldsymbol{R}_0^*(\theta)]_{i,j} > 0$ if and only if $[\boldsymbol{R}_0^*(1)] > 0$. Note also that $[\boldsymbol{R}_0^*(1)]_{i,j}$ represents the expected number of visits to phase j during the first passage time from state $(0, i)$ to level zero. Thus $\boldsymbol{R}_0^*(\theta) \ge \boldsymbol{O}$ has no zero column due to Assumption 1.1 (a). Consequently, $\boldsymbol{x}(0)\boldsymbol{R}_0^*(\theta)\boldsymbol{s}(\theta) > 0$ because $\boldsymbol{x}(0) > \mathbf{0}$. □

Remark 3.2 In Theorem 3.5 in [7], (46) is assumed in order to prove $c(1) > 0$.

Lemma 3.4 ensures that (39) holds. Thus substituting (44) into (39) yields

$$\overline{\boldsymbol{x}}(k) = \theta^{-k} \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^k} c(\omega_\tau^\nu) \cdot \boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1} + O((\theta + \varepsilon_0)^{-k})\boldsymbol{e}^{\mathrm{t}}. \tag{48}$$

According to (40), the dominant term of (48) is positive for any $k = 0, 1, \dots$. Letting $k = n\tau + l$ ($l = 0, 1, \dots, \tau - 1$, $n = 0, 1, \dots$) in (48), we readily obtain the following theorem.

Theorem 3.1 *Suppose Assumptions 1.1–1.3 hold and $\theta < r_B$. Then*

$$\overline{\boldsymbol{x}}(n\tau + l) = \theta^{-n\tau - l}\boldsymbol{c}_l + O((\theta + \varepsilon_0)^{-(n\tau + l)})\boldsymbol{e}^{\mathrm{t}}, \qquad l = 0, 1, \dots, \tau - 1,$$

where

$$\boldsymbol{c}_l = \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^l} \frac{[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta\omega_\tau^\nu) - \boldsymbol{x}(1)\boldsymbol{A}(0)]\,\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta)}{(\theta\omega_\tau^\nu - 1)\{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta} - 1\}} \cdot \boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1} > \mathbf{0}. \tag{49}$$

Remark 3.3 Theorem 3.1 and Propositions 2.6 and 2.12 imply that $\boldsymbol{c}_l$ ($l = 0, 1, \dots, \tau - 1$) in (49) must be equal to $\boldsymbol{d}_l$ in Proposition 2.6. However in the case of $\tau = h \ge 2$, it seems difficult to demonstrate $\boldsymbol{c}_l = \boldsymbol{d}_l$ ($l = 0, 1, \dots, \tau - 1$), because these two expressions are completely different. On the other hand, in the case of $\tau = h = 1$, we can readily confirm $\boldsymbol{c}_0$ is equal to the constant vector on the right hand side of (23) (see Proposition 2.7).

Corollary 3.1 *Suppose Assumptions 1.1–1.3 hold and $\theta < r_B$. If $\boldsymbol{C}(0) = \boldsymbol{A}(0)$, then $\boldsymbol{c}_l$ ($l = 0, 1, \dots, \tau - 1$) is given by*

$$\boldsymbol{c}_l = \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^l} \frac{\boldsymbol{x}(0)\,[\boldsymbol{B}^*(\theta\omega_\tau^\nu) + \boldsymbol{B}(0) - \boldsymbol{I}]\,\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta)}{(\theta\omega_\tau^\nu - 1)\{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta} - 1\}} \cdot \boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}. \tag{50}$$

In addition, if $\boldsymbol{B}(k) = \boldsymbol{A}(k)$ for all $k = 0, 1, \dots$, then (50) is reduced to

$$\boldsymbol{c}_l = \sum_{\nu=0}^{\tau-1} \frac{1}{(\omega_\tau^\nu)^l} \frac{(1-\rho)\boldsymbol{g}\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta)}{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta} - 1} \cdot \boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_\tau^\nu)^{-1}, \qquad l = 0, 1, \dots, \tau-1. \tag{51}$$

Proof. When $\boldsymbol{C}(0) = \boldsymbol{A}(0)$, $\boldsymbol{x}(1)\boldsymbol{A}(0) = \boldsymbol{x}(0) - \boldsymbol{x}(0)\boldsymbol{B}(0)$. Substituting this into (49) yields (50). We now suppose that $\boldsymbol{C}(0) = \boldsymbol{A}(0)$ and $\boldsymbol{B}(k) = \boldsymbol{A}(k)$ for all $k = 0, 1, \dots$. We then have

$$\begin{aligned}
[\boldsymbol{B}^*(\theta\omega_\tau^\nu) + \boldsymbol{B}(0) - \boldsymbol{I}]\, \boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta) &= [\boldsymbol{A}^*(\theta\omega_\tau^\nu) - \boldsymbol{I}]\, \boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta) \\
&= [(\theta\omega_\tau^\nu)\boldsymbol{\Gamma}_A^*(\theta\omega_\tau^\nu) - \boldsymbol{I}]\, \boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta) \\
&= (\theta\omega_\tau^\nu - 1)\boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta),
\end{aligned} \tag{52}$$

where the second equality follows from (16) and the third one follows from $\boldsymbol{v}(\theta\omega_\tau^\nu) = \boldsymbol{\Delta}_M(\omega_\tau^\nu)\boldsymbol{v}(\theta)$ and $\delta(\boldsymbol{\Gamma}_A(\theta\omega_\tau^\nu)) = 1$ (see (28) and Proposition 2.11). Substituting (52) and $\boldsymbol{x}(0) = (1-\rho)\boldsymbol{g}$ (see, e.g., Takine [25]) into (50), we have (51). □

We close this subsection by discussing the period of the geometric asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$. It should be noted that Theorem 3.1 does not necessarily show that the period in the geometric asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ is equal to τ. This is because $c(\omega_\tau^\nu)$ ($\nu = 1, 2, \dots, \tau-1$) may be equal to zero i.e., $z = \theta\omega_\tau^\nu$ ($\nu = 1, 2, \dots, \tau-1$) may not be a pole of $[\overline{\boldsymbol{x}}^*(z)]_j$'s ($j \in \mathbb{M}$). Let $\mathcal{P}_A$ denote the set of poles of $[\overline{\boldsymbol{x}}^*(z)]_j$'s on $\{z \in \mathbb{C}; |z| = \theta\}$. Since $c(1) > 0$ (see Lemma 3.4), $\mathcal{P}_A$ includes θ and thus

$$\mathcal{P}_A = \{\theta\omega_\tau^\nu; \nu \in \mathbb{H}\},$$

where $\mathbb{H} = \{0\} \cup \{\nu \in \{1, 2, \dots, \tau-1\}; c(\omega_\tau^\nu) \neq 0\}$. Let $\tau' = \tau/\gcd\{\nu \in \{\tau\} \cup \mathbb{H} \setminus \{0\}\}$. Let H denote the number of elements in $\mathbb{H}$. It is easy to see that there exists H nonnegative integers, ν_m's ($m = 0, 1, \dots, H-1$), such that $0 = \nu_0 < \nu_1 < \cdots < \nu_{H-1} \le \tau' - 1$ and

$$\mathcal{P}_A = \{\theta\omega_{\tau'}^{\nu_m}; m = 0, 1, \dots, H-1\}. \tag{53}$$

Note that since $\overline{\boldsymbol{x}}(k) > \boldsymbol{0}$ ($k = 0, 1, \dots$), each $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$) has pairs of complex conjugate poles and therefore

$$\omega_{\tau'}^{\nu_m}\omega_{\tau'}^{\nu_{H-m}} = 1, \qquad m = 1, 2, \dots, \lfloor (H-1)/2 \rfloor. \tag{54}$$

It follows from (48) and $c(\omega_\tau^\nu) = 0$ for $\nu \notin \mathbb{H}$ that

$$\overline{\boldsymbol{x}}(k) = \theta^{-k} \sum_{m=0}^{H-1} \frac{1}{(\omega_{\tau'}^{\nu_m})^k} c(\omega_{\tau'}^{\nu_m}) \cdot \boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_{\tau'}^{\nu_m})^{-1} + O((\theta+\varepsilon_0)^{-k})\boldsymbol{e}^{\mathrm{t}}.$$

Letting $k = n\tau' + l$ ($l = 0, 1, \dots, \tau' - 1$, $n = 0, 1, \dots$) in the above equation yields

$$\overline{\boldsymbol{x}}(n\tau' + l) = \theta^{-n\tau' - l}\boldsymbol{c}_l' + O((\theta + \varepsilon_0)^{-(n\tau'+l)})\boldsymbol{e}^{\rm t}, \qquad l = 0, 1, \dots, \tau' - 1,$$

where

$$\boldsymbol{c}_l' = \sum_{m=0}^{H-1} \frac{1}{(\omega_{\tau'}^{\nu_m})^l} \frac{[\boldsymbol{x}(0)\boldsymbol{B}^*(\theta\omega_{\tau'}^{\nu_m}) - \boldsymbol{x}(1)\boldsymbol{A}(0)]\, \boldsymbol{\Delta}_M(\omega_{\tau'}^{\nu_m})\boldsymbol{v}(\theta)}{(\theta\omega_{\tau'}^{\nu_m} - 1)\{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta} - 1\}} \cdot \boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\omega_{\tau'}^{\nu_m})^{-1} > \boldsymbol{0}.$$

As a result, the period in the geometric asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ is equal to τ'.

3.2 Case of $\theta > r_B$

For simplicity, we denote, by $C(\zeta, r)$, the circle $\{z \in \mathbb{C}; |z - \zeta| = r\}$ in the complex plane, where $\zeta \in \mathbb{C}$ and $r \ge 0$. In this subsection, we make the following assumption.

Assumption 3.1 (a) $\boldsymbol{B}^*(z)$ is meromorphic in an open set containing the domain $\{z \in \mathbb{C}; |z| \le r_B\}$[‡2], and (b) there exists some positive integer m_B and some finite nonnegative matrix $\widetilde{\boldsymbol{B}}(r_B)$ such that

$$\lim_{z \to r_B} \left(1 - \frac{z}{r_B}\right)^{m_B} \boldsymbol{B}^*(z) = \widetilde{\boldsymbol{B}}(r_B) \ne \boldsymbol{O}.$$

Remark 3.4 From the definition, $\boldsymbol{B}^*(z)$ is holomorphic in the domain $\{z \in \mathbb{C}; |z| < r_B\}$. Thus Assumption 3.1 is an additional condition for the behavior of $\boldsymbol{B}^*(z)$ on the convergence radius. In fact, Assumption 3.1 shows that $\boldsymbol{B}^*(z)$ is holomorphic on $C(0, r_B)$ except for its poles.

It follows from Lemma A.1 that under Assumption 3.1, any pole of $[\boldsymbol{B}^*(z)]_{i,j}$'s on $C(0, r_B)$ is of order less than or equal to m_B. Thus we assume the following.

Assumption 3.2 There exist exactly N ($N \in \mathbb{N}$) complex numbers ζ_n's $(n = 0, 1, \dots, N-1)$ on $C(0, 1)$ such that $0 = \arg(\zeta_0) < \arg(\zeta_1) < \cdots < \arg(\zeta_{N-1}) < 2\pi$ and

$$\lim_{z \to r_B\zeta_n} \left(1 - \frac{z}{r_B\zeta_n}\right)^{m_B} \boldsymbol{B}^*(z) = \widetilde{\boldsymbol{B}}(r_B\zeta_n), \qquad n = 0, 1, \dots, N - 1,$$

where $\widetilde{\boldsymbol{B}}(r_B\zeta_n)$ is some finite non-zero matrix.

[‡2] In the published version, it is assumed that "$\boldsymbol{B}^*(z)$ is meromorphic in the domain $\{z \in \mathbb{C}; |z| \le r_B\}$". However, by the definition of meromorphicness, the revised description is more appropriate. The same is true of Assumption A.1.

Remark 3.5 Since $\boldsymbol{B}(k) \ge \boldsymbol{O}$ for all $k = 1, 2, \dots$,

$$[\boldsymbol{B}^*(\mathrm{conj}(z))]_{i,j} = \mathrm{conj}\left([\boldsymbol{B}^*(z)]_{i,j}\right), \qquad i, j \in \mathbb{M},$$

where $\mathrm{conj}(z)$ ($z \in \mathbb{C}$) denotes the complex conjugate of z. Thus if $z = r_B\zeta$ is a pole of $[\boldsymbol{B}^*(z)]_{i,j}$, so is $z = r_B \mathrm{conj}(\zeta)$. This fact implies that

$$\zeta_n \zeta_{N-n} = 1, \qquad n = 1, 2, \dots, \lfloor (N-1)/2 \rfloor, \tag{55}$$

and therefore for any $i, j \in \mathbb{M}$,

$$\left[\widetilde{\boldsymbol{B}}(r_B \zeta_{N-n})\right]_{i,j} = \mathrm{conj}\left([\widetilde{\boldsymbol{B}}(r_B\zeta_n)]_{i,j}\right), \qquad n = 1, 2, \dots, \lfloor (N-1)/2 \rfloor.$$

Lemma 3.1 implies that if $\theta > r_B$, then $\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)$ is nonsingular in the domain $\{z \in \mathbb{C}; |z| \le r_B\}$ and therefore $\overline{\boldsymbol{x}}^*(z)$ in (34) is holomorphic in the same domain. It thus follows from Assumptions 3.1 and 3.2 that for $n = 0, 1, \dots, N-1$,

$$\lim_{z \to r_B\zeta_n} \left(1 - \frac{z}{r_B \zeta_n}\right)^{m_B} \overline{\boldsymbol{x}}^*(z) = \frac{\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(r_B\zeta_n)}{r_B\zeta_n - 1} \left[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(r_B\zeta_n)\right]^{-1}. \tag{56}$$

Note here that for each $n = 0, 1, \dots, N-1$, $z = r_B\zeta_n$ is an m_Bth order pole of $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$) if and only if

$$\left[\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(r_B\zeta_n)\left[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(r_B\zeta_n)\right]^{-1}\right]_j \neq 0.$$

Note also that

$$\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(r_B)\left[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(r_B)\right]^{-1} > \boldsymbol{0},$$

which follows from $\boldsymbol{x}(0) > \boldsymbol{0}$, $\widetilde{\boldsymbol{B}}(r_B) \ge \boldsymbol{O}, \neq \boldsymbol{O}$ and $[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(r_B)]^{-1} > \boldsymbol{O}$. Therefore we obtain the following lemma.

Lemma 3.5 *If Assumptions 1.1–1.3 and 3.1 hold and $\theta > r_B$, then $z = r_B$ is an m_Bth order pole of $[\overline{\boldsymbol{x}}^*(z)]_j$ for any $j \in \mathbb{M}$.*

Remark 3.6 Suppose all the conditions of Lemma 3.5 hold except Assumption 1.3. Then in this case, we can readily show that if $r_A > r_B$ (instead of $\theta > r_B$), $\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(r_B)\left[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(r_B)\right]^{-1} > \boldsymbol{0}$ and thus $z = r_B$ is an m_Bth order pole of $[\overline{\boldsymbol{x}}^*(z)]_j$ for any $j \in \mathbb{M}$.

From (56) and Theorem A.1, we readily obtain the following result.

Theorem 3.2 *Suppose Assumptions 1.1–1.3, 3.1 and 3.2 hold and $\theta > r_B$. We then have*

$$\overline{\boldsymbol{x}}(k) = \frac{k^{m_B-1}}{(m_B-1)!}\frac{1}{r_B^k}\boldsymbol{\xi}(k) + \begin{cases} O((r_B+\varepsilon_0)^{-k})\boldsymbol{e}^{\mathrm{t}}, & m_B = 1, \\ O(k^{m_B-2}r_B^{-k})\boldsymbol{e}^{\mathrm{t}}, & m_B \ge 2, \end{cases} \tag{57}$$

where

$$\boldsymbol{\xi}(k) = \sum_{n=0}^{N-1} \frac{1}{\zeta_n^k} \frac{\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(r_B\zeta_n)}{r_B\zeta_n - 1} \left[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(r_B\zeta_n)\right]^{-1}.$$

Further $\limsup_{k\to\infty} \boldsymbol{\xi}(k) > \mathbf{0}$ *and* $\boldsymbol{\xi}(k) \ge \mathbf{0}$ *for all* $k = 0, 1, \dots$.

Remark 3.7 According to Theorem A.1 (d), if $(\arg \zeta_n)/\pi$ is rational for any $n = 0, 1, \dots, N-1$, then $\boldsymbol{\xi}(k) > \mathbf{0}$ for any $k = 0, 1, \dots$.

Corollary 3.2 *Suppose Assumptions 1.1–1.3 and 3.1 hold and* $\theta > r_B$*. If* $N = 1$*, then*

$$\overline{\boldsymbol{x}}(k) = \frac{\boldsymbol{x}(0)\overline{\boldsymbol{B}}(k)}{r_B - 1}[\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(r_B)]^{-1} + \begin{cases} O((r_B + \varepsilon_0)^{-k})\boldsymbol{e}^{\mathrm{t}}, & m_B = 1, \\ O(k^{m_B-2} r_B^{-k})\boldsymbol{e}^{\mathrm{t}}, & m_B \ge 2. \end{cases} \tag{58}$$

The dominant term on the right hand side of (58) is a positive vector for any $k = 0, 1, \dots$.

We now mention the case where Assumption 1.3 does not hold, i.e., there does not exist θ such that $1 < \theta < r_A$ and $\theta = \delta(\boldsymbol{A}^*(\theta))$. In this case, $\det(\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = 0$ has no root in the domain $\{z \in \mathbb{C}; 1 < |z| < r_A\}$, and thus if Assumption 3.1 holds and $r_A > r_B$, then $z = r_B$ is a dominant pole with order m_B of $[\overline{\boldsymbol{x}}^*(z)]_j$ for any $j \in \mathbb{M}$ (see Remark 3.6). Therefore we have the following result.

Theorem 3.3 *Suppose there does not exist* θ *such that* $1 < \theta < r_A$ *and* $\theta = \delta(\boldsymbol{A}^*(\theta))$*. Further suppose Assumptions 1.1, 1.2, 3.1 and 3.2 are satisfied and* $r_A > r_B$*. Then (57) holds, and (58) does if* $N = 1$*.*

3.3 Case of $\theta = r_B$

This subsection considers the case of $\theta = r_B$ under Assumptions 1.1–1.3, 3.1 and 3.2. Lemmas 3.4 and 3.5 show that $z = \theta$ (resp. r_B) is a simple pole (resp. an m_Bth order pole) of each $[\overline{\boldsymbol{x}}^*(z)]_j$ ($j \in \mathbb{M}$). Thus if $\theta = r_B$, $z = \theta$ $(= r_B)$ is the $(m_B + 1)$st order pole of each $[\overline{\boldsymbol{x}}^*(z)]_j$ and its dominant poles are included in $\mathcal{P} \triangleq \mathcal{P}_A \cap \mathcal{P}_B$, where $\mathcal{P}_A$ is given in (53) and $\mathcal{P}_B = \{\theta\zeta_n; n = 0, 1, \dots, N-1\}$. Let L denote the number of elements in $\mathcal{P}$. Let η_m ($m = 0, 1, \dots, L-1$)'s denote L nonnegative integers such that $0 = \eta_0 < \eta_1 < \cdots < \eta_{L-1} \le \tau' - 1$ and

$$\mathcal{P} = \{\theta\omega_{\tau'}^{\eta_m}; m = 0, 1, \dots, L-1\}.$$

Let $\widehat{\tau}$ denote

$$\widehat{\tau} = \tau' / \gcd\{\eta_1, \eta_2, \dots, \eta_{L-1}, \tau'\}.$$

For simplicity, let $\widehat{\omega}_m = (\omega_{\widehat{\tau}})^{\widehat{\eta}_m}$ ($m = 0, 1, \dots, L-1$), where $\widehat{\eta}_m = (\widehat{\tau}/\tau')\eta_m$. It then follows that $\mathcal{P} = \{\theta\widehat{\omega}_m; m = 0, 1, \dots, L-1\}$. Note here that $\widehat{\omega}_m\widehat{\omega}_{L-m} = 1$ for $m = 1, 2, \dots, \lfloor (L-1)/2 \rfloor$ due to (54) and (55).

Theorem 3.4 *If Assumptions 1.1–1.3, 3.1 and 3.2 hold and $\theta = r_B$, then*

$$\overline{\boldsymbol{x}}(n\widehat{\tau}+l) = \frac{(n\widehat{\tau}+l)^{m_B}}{m_B!}\frac{1}{\theta^{n\widehat{\tau}+l}}\widehat{\boldsymbol{c}}_l + O((n\widehat{\tau}+l)^{m_B-1}\theta^{-(n\widehat{\tau}+l)})\boldsymbol{e}^{\mathrm{t}}, \tag{59}$$

where

$$\widehat{\boldsymbol{c}}_l = \sum_{m=0}^{L-1}\frac{1}{(\widehat{\omega}_m)^l}\frac{\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(\theta\widehat{\omega}_m)\boldsymbol{\Delta}_M(\widehat{\omega}_m)\boldsymbol{v}(\theta)}{(\theta\widehat{\omega}_m-1)\{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta}-1\}}\cdot\boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\widehat{\omega}_m)^{-1} > \boldsymbol{0}.$$

Remark 3.8 Theorem 3.4 shows that the period in the geometric asymptotics of $\{\overline{\boldsymbol{x}}(k)\}$ is divisor of $\widehat{\tau}$. It seems difficult to say more about the period in the general setting.

Proof of Theorem 3.4 Recall that $z=\theta$ is an (m_B+1)st order pole of $[\overline{\boldsymbol{x}}^*(z)]_j$ for any $j \in \mathbb{M}$ and that $z=\theta\widehat{\omega}_m$ $(m=1,2,\dots,L-1)$ can be minimum-modulus poles of order m_B+1. It then follows from Theorem A.1 that

$$\overline{\boldsymbol{x}}(k) = \frac{k^{m_B}}{m_B!}\frac{1}{\theta^k}\sum_{m=0}^{L-1}\frac{1}{(\widehat{\omega}_m)^k}\lim_{z\to\theta\widehat{\omega}_m}\left(1-\frac{z}{\theta\widehat{\omega}_m}\right)^{m_B+1}\overline{\boldsymbol{x}}^*(z) + O(k^{m_B-1}\theta^{-k})\boldsymbol{e}^{\mathrm{t}}, \tag{60}$$

where the dominant term is positive for any $k=0,1,\dots$. Applying Assumption 3.2 and Lemma 3.3 to (34) yield

$$\begin{aligned}
&\lim_{z\to\theta\widehat{\omega}_m}\left(1-\frac{z}{\theta\widehat{\omega}_m}\right)^{m_B+1}\overline{\boldsymbol{x}}^*(z)\\
&\quad= \lim_{z\to\theta\widehat{\omega}_m}\left(1-\frac{z}{\theta\widehat{\omega}_m}\right)^{m_B}\frac{\boldsymbol{x}(0)\boldsymbol{B}^*(z)-\boldsymbol{x}(1)\boldsymbol{A}(0)}{z-1}\\
&\qquad\cdot\lim_{z\to\theta\widehat{\omega}_m}\left(1-\frac{z}{\theta\widehat{\omega}_m}\right)[\boldsymbol{I}-\boldsymbol{\Gamma}_A^*(z)]^{-1}\\
&\quad= \frac{\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(\theta\widehat{\omega}_m)}{\theta\widehat{\omega}_m-1}\frac{\boldsymbol{\Delta}_M(\widehat{\omega}_m)\boldsymbol{v}(\theta)\cdot\boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\widehat{\omega}_m)^{-1}}{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta}-1},
\end{aligned}$$

from which and (60) we obtain

$$\begin{aligned}
\overline{\boldsymbol{x}}(k) &= \frac{k^{m_B}}{m_B!}\frac{1}{\theta^k}\sum_{m=0}^{L-1}\frac{1}{(\widehat{\omega}_m)^k}\frac{\boldsymbol{x}(0)\widetilde{\boldsymbol{B}}(\theta\widehat{\omega}_m)}{\theta\widehat{\omega}_m-1}\frac{\boldsymbol{\Delta}_M(\widehat{\omega}_m)\boldsymbol{v}(\theta)\cdot\boldsymbol{\mu}(\theta)\boldsymbol{\Delta}_M(\widehat{\omega}_m)^{-1}}{(\mathrm{d}/\mathrm{d}y)\delta(\boldsymbol{A}^*(y))|_{y=\theta}-1}\\
&\quad+ O(k^{m_B-1}\theta^{-k})\boldsymbol{e}^{\mathrm{t}}.
\end{aligned}$$

As a result, we obtain (59) by letting $k=n\widehat{\tau}+l$ ($n=0,1,\dots$, $l=0,1,\dots,\widehat{\tau}-1$) in the above equation and using $(\widehat{\omega}_m)^{n\widehat{\tau}+l}=(\widehat{\omega}_m)^l$. □

3.4 Remarks

One of the referees informed the authors that a parallel research by Dr. Tai [24] was open to the public after the submission of this paper. The research is on the light-tailed asymptotics of the stationary probability vectors $\{\boldsymbol{x}(k)\}$ of the Markov chain of GI/G/1 type. Tai derives the decay rate of $\{\boldsymbol{x}(k)\}$ in a weak sense, i.e., $-\log \lim_{k\to\infty} \left([\boldsymbol{x}(k)]_j\right)^{1/k}$, and also presents several conditions under which $\{\boldsymbol{x}(k)\}$ is asymptotically geometric, or light-tailed but not exactly geometric, assuming the aperiodicity of the MAdP driven by the transition block matrices in the non-boundary levels. As with this paper, Tai's research includes the case where the jumps from the boundary level have the dominant impact on the decay of the stationary tail probability vectors.

A Tail Asymptotics of Nonnegative Sequences

Let $\{x_k; k = 0, 1, \dots\}$ denote a sequence of nonnegative numbers, an infinite number of which are positive. Let σ denote

$$\sigma = \sup \left\{ |z|; \sum_{k=0}^{\infty} x_k z^k < \infty, z \in \mathbb{C} \right\},$$

which is called the convergence radius of the power series. Let $f(z)$ denote the generating function of $\{x_k; k = 0, 1, \dots\}$. We then have

$$f(z) = \sum_{k=0}^{\infty} x_k z^k, \qquad |z| < \sigma. \tag{61}$$

Further by definition, $f(z)$ is holomorphic inside the convergence radius.

In what follows, we make the following assumption.

Assumption A.1 $f(z)$ is meromorphic in an open set containing the domain $\{z \in \mathbb{C}; |z| \le \sigma\}$[‡3], and the point $z = \sigma$ is an $\breve{m}$th pole of $f(z)$, where $\breve{m}$ is some finite positive integer.

Lemma A.1 *Under Assumption A.1, any pole of $f(z)$ on $C(0,\sigma)$ is of order less than or equal to $\breve{m}$.*

Proof. We define $g(z)$ as

$$g(z) = f(z) \left(1 - \frac{z}{\sigma}\right)^{\breve{m}}.$$

[‡3] In the published version, it is assumed that "$f(z)$ is meromorphic in the domain $\{z \in \mathbb{C}; |z| \le \sigma\}$".

From (61), we have for any $\varepsilon > 0$,

$$g(\sigma - \varepsilon) = f(\sigma - \varepsilon)\left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}} = \sum_{k=0}^{\infty} x_k(\sigma-\varepsilon)^k\left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}}. \tag{62}$$

It thus follows from (61) and (62) that for any $\omega_* \in \mathbb{C}$ such that $|\omega_*| = 1$ and $\omega_* \neq 1$,

$$\begin{aligned}\liminf_{\substack{z=(\sigma-\varepsilon)\omega_*\\ \varepsilon\downarrow 0}} \left| f(z)\left(1-\frac{z}{\sigma\omega_*}\right)^{\breve{m}}\right| &= \liminf_{\varepsilon\downarrow 0}\left|\sum_{k=0}^{\infty} x_k(\sigma-\varepsilon)^k(\omega_*)^k\left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}}\right| \\ &\le \limsup_{\varepsilon\downarrow 0}\sum_{k=0}^{\infty} x_k(\sigma-\varepsilon)^k\left(\frac{\varepsilon}{\sigma}\right)^{\breve{m}} \\ &= \limsup_{\varepsilon\downarrow 0} g(\sigma-\varepsilon) = g(\sigma) < \infty, \end{aligned} \tag{63}$$

where the last inequality holds because $g(z)$ is holomorphic in some neighborhood of $z = \sigma$. Let $\breve{m}_*$ denote

$$\breve{m}_* = \inf\left\{m \in \mathbb{N}\cup\{0\}; \lim_{z\to\sigma\omega_*}\left|f(z)\left(1-\frac{z}{\sigma\omega_*}\right)^{m}\right| < \infty\right\},$$

where $f(z)(1 - z/(\sigma\omega_*))^m$ is meromorphic in the domain $\{z \in \mathbb{C}; |z| \le \sigma\}$ for $m = 0, 1, \dots$. Thus, if $\breve{m}_* > \breve{m}$, we have

$$\liminf_{\substack{z=(\sigma-\varepsilon)\omega_*\\ \varepsilon\downarrow 0}} \left| f(z)\left(1-\frac{z}{\sigma\omega_*}\right)^{\breve{m}}\right| \ge \liminf_{z\to\sigma\omega_*}\left| f(z)\left(1-\frac{z}{\sigma\omega_*}\right)^{\breve{m}}\right| = \infty,$$

which contradicts (63). As a result, $\breve{m}_* \le \breve{m}$, which implies that this lemma is true. □

According to Lemma A.1, we introduce the following definition.

Definition A.1 Under Assumption A.1, *a dominant pole* of $f(z)$ is a pole that is located on its convergence circle $C(0, \sigma)$ and is of the same order as that of pole $z = \sigma$. Thus the order of any dominant pole of $f(z)$ is equal to $\breve{m}$.

We make the following assumption, in addition to Assumption A.1.

Assumption A.2 There exist exactly P ($P \ge 1$) dominant poles, σ_j's ($j = 0, 1, \dots, P-1$), of $f(z)$, where $\sigma_0 = \sigma$ and $0 = \arg\sigma_0 < \arg\sigma_1 < \cdots < \arg\sigma_{P-1} < 2\pi$.

Remark A.1 Since $f(z)$ is the generating function of the nonnegative sequence $\{x_k\}$, the set $\{\sigma_j; j = 0, 1, \dots, P-1\}$ consists of one or two real numbers and $\lfloor (P-1)/2 \rfloor$ pairs of conjugate complex numbers. Therefore $\sigma_j\sigma_{P-j} = \sigma^2$ for $j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor$.

Theorem A.1 *Suppose Assumptions A.1 and A.2 hold, and let* $a_{1,k} = 1/(\sigma+\varepsilon_0)^k$ *(*$k = 0, 1, \dots$*), where* $\varepsilon_0 > 0$ *is a sufficiently small number; and for* $m = 2, 3, \dots$*,* $a_{m,k} = k^{m-2}/\sigma^k$ *(*$k = 0, 1, \dots$*). Then the following are true.*

(a) The sequence $\{x_k; k = 0, 1, \dots\}$ *satisfies*

$$\begin{aligned} x_k &= \binom{k + \breve{m} - 1}{\breve{m} - 1} \frac{1}{\sigma^k} \xi_k + O(a_{\breve{m},k}) \\ &= \frac{k^{\breve{m}-1}}{(\breve{m} - 1)!} \frac{1}{\sigma^k} \xi_k + O(a_{\breve{m},k}), \end{aligned} \tag{64}$$

where

$$\xi_k = \sum_{j=0}^{P-1} \left(\frac{\sigma}{\sigma_j} \right)^k \lim_{z \to \sigma_j} \left(1 - \frac{z}{\sigma_j} \right)^{\breve{m}} f(z). \tag{65}$$

(b) $\limsup_{k \to \infty} \xi_k > 0$.

(c) $\xi_k \geq 0$ *for all* $k = 0, 1, \dots$.

(d) In addition, if $\{x_k\}$ *is eventually*[‡4] *nonincreasing and* $(\arg \sigma_j)/\pi$ *is a rational number for any* $j = 0, 1, \dots, P-1$*, then* $\xi_k > 0$ *for all* $k = 0, 1, \dots$.

Proof. See Appendix C.5. □

Remark A.2 A result similar to the statement (a) is given in Theorem 5.2.1 in [27]. Further when $P = 1$, (65) is reduced to eq. (2) at p. 238 in [5].

Remark A.3 Suppose the candidates for the dominant poles are σ_j's ($j = 0, 1, \dots, P-1$) and at least one of them is indeed a dominant pole (according to Assumption A.1, $z = \sigma$ is a dominant pole). For σ_j not a dominant pole, we have

$$\lim_{z \to \sigma_j} \left(1 - \frac{z}{\sigma_j} \right)^{\breve{m}} f(z) = 0.$$

Thus the statements (a)–(d) of Theorem A.1 still hold, though the right hand side of (65) may include some null terms.

B Period of Markov Additive Processes

This appendix summarizes fundamental results of the period of MAdPs. In fact, most of the results described here are already implied in [2, 23], though that is

[‡4] In this revised version, we add "eventually" in order to (slightly) strengthen the statement.

done in not an accessible way. Further a MAdP related to the Markov chain of M/G/1 type (and slightly more general one) are discussed in [8].

We consider a MAdP $\{(\Gamma_n, J_n); n = 0, 1, \dots\}$, where the level variable Γ_n takes a value in $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and the phase variable J_n takes a value in $\mathbb{J} \triangleq \{1, 2, \dots, J\}$. Let $\mathbf{\Gamma}(k)$ $(k \in \mathbb{Z})$ denote a $J \times J$ matrix whose (i, j)th $(i, j \in \mathbb{J})$ element represents

$$\Pr[\Gamma_{n+1} = k_0 + k, J_{n+1} = j \mid \Gamma_n = k_0, J_n = i],$$

for any fixed $k_0 \in \mathbb{Z}$. For simplicity, we denote the MAdP $\{(\Gamma_n, J_n); n = 0, 1, \dots\}$ with kernel $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$ by MAdP $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$. For any two states (k_1, j_1) and (k_2, j_2) in $\mathbb{Z} \times \mathbb{J}$, we write $(k_1, j_1) \to (k_2, j_2)$ when there exists a path from (k_1, j_1) to (k_2, j_2) with some positive probability.

Assumption B.1

(a) $\mathbf{\Gamma} \triangleq \sum_{k \in \mathbb{Z}} \mathbf{\Gamma}(k)$ is irreducible.

(b) For each $j \in \mathbb{J}$, there exists a nonzero integer k_j such that $(0, j) \to (k_j, j)$.

Let $\mathbb{K}_j$ $(j \in \mathbb{J})$ denote

$$\mathbb{K}_j = \{k \in \mathbb{Z} \backslash \{0\};\ (0, j) \to (k, j)\},$$

which is well-defined under Assumption B.1.

Lemma B.1 *Under Assumption B.1, let $d_j = \gcd\{k \in \mathbb{K}_j\}$ for $j \in \mathbb{J}$. Then d_j's $(j \in \mathbb{J})$ are all identical.*

Proof. See Appendix C.6. □

Definition B.1 According to Lemma B.1, we write d to represent d_j's and refer to the constant d as the period of MAdP $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$.

We choose a state $i_0 \in \mathbb{J}$ and then define $\mathbb{J}_0^{(i_0)}$ as

$$\mathbb{J}_0^{(i_0)} = \{j \in \mathbb{J};\ (0, i_0) \to (k, j),\ k \equiv 0 \pmod{d}\}.$$

We also define $\mathbb{J}_m^{(i_0)}$ $(m = 1, 2, \dots, d-1)$ as

$$\mathbb{J}_m^{(i_0)} = \{j \in \mathbb{J};\ (0, i_0) \to (k, j),\ k \equiv m \pmod{d}\}.$$

Since $\mathbf{\Gamma}$ is irreducible, each $j \in \mathbb{J}$ must belong to at least one of $\{\mathbb{J}_m^{(i_0)}; m = 0, 1, \dots, d-1\}$. Further for any $i \in \mathbb{J}_m^{(i_0)}$,

$$(0, i) \to (k, i_0) \text{ only if } k \equiv -m \pmod{d},$$

which implies that $\mathbb{J}_{m_1}^{(i_0)} \cap \mathbb{J}_{m_2}^{(i_0)} = \emptyset$ for $m_1 \not\equiv m_2$ (if not, it would hold that $(0, i_0) \to (k, i_0)$ for some $k \equiv m_1 - m_2 \not\equiv 0 \pmod{d}$). Thus $\mathbb{J}_0^{(i_0)} + \mathbb{J}_1^{(i_0)} + \cdots + \mathbb{J}_{d-1}^{(i_0)} = \mathbb{J}$ and there exists a function[‡5] q_0 from $\mathbb{J}$ to $\{0, 1, \dots, d-1\}$ such that $j \in \mathbb{J}_{q_0(j)}^{(i_0)}$. It follows from the definition of $\mathbb{J}_m^{(i_0)}$'s that

$$[\mathbf{\Gamma}(k)]_{i,j} > 0 \text{ only if } k \equiv q_0(j) - q_0(i) \pmod{d}.$$

As a result, we obtain the following result.

Lemma B.2 *Under Assumption B.1, the period d is the largest positive integer such that*

$$[\mathbf{\Gamma}(k)]_{i,j} > 0 \textit{ only if } k \equiv q(j) - q(i) \pmod{d}, \tag{66}$$

where q is some function[‡6] *from $\mathbb{J}$ to $\{0, 1, \dots, d-1\}$. Further let $\mathbb{J}_m = \{j \in \mathbb{J}; q(j) = m\}$ for $m = 0, 1, \dots, d-1$. Then $\mathbb{J}_m$'s ($m = 0, 1, \dots, d-1$) are disjoint each other and $\mathbb{J}_0 + \mathbb{J}_1 + \cdots + \mathbb{J}_{d-1} = \mathbb{J}$.*

In the rest of this section, we discuss the relationship between the period d of MAdP $\{\mathbf{\Gamma}(k); k \in \mathbb{Z}\}$ and the eigenvalues of the generating function $\mathbf{\Gamma}^*(z)$ defined by $\sum_{k\in\mathbb{Z}} z^k \mathbf{\Gamma}(k)$. Let $\mathbf{\Delta}(z)$ denote a $J \times J$ diagonal matrix whose jth diagonal element is equal to $z^{-q(j)}$. It then follows from (66) that

$$\mathbf{\Gamma}^*(z) = \mathbf{\Delta}(z)\mathbf{\Lambda}^*(z^d)\mathbf{\Delta}(z)^{-1} = \mathbf{\Delta}(z/|z|)\mathbf{\Lambda}^*(z^d)\mathbf{\Delta}(z/|z|)^{-1}, \tag{67}$$

where $\mathbf{\Lambda}^*(z)$ denotes a $J \times J$ matrix whose (i, j)th element is given by

$$[\mathbf{\Lambda}^*(z)]_{i,j} = \sum_{n\in\mathbb{Z}} z^n [\mathbf{\Gamma}(nd + q(j) - q(i))]_{i,j}.$$

Let $\boldsymbol{\gamma}(z)$ and $\boldsymbol{g}(z)$ denote left- and right-eigenvectors of $\mathbf{\Gamma}^*(z)$ corresponding to eigenvalue $\delta(\mathbf{\Gamma}^*(z))$, normalized such that

$$\boldsymbol{\gamma}(z)\mathbf{\Delta}(z/|z|)\boldsymbol{e} = 1, \qquad \boldsymbol{\gamma}(z)\boldsymbol{g}(z) = 1. \tag{68}$$

We then have the following lemma.

Lemma B.3 *Suppose Assumption B.1 holds, and let $I_\gamma = \{y > 0; \sum_{k\in\mathbb{Z}} y^k \mathbf{\Gamma}(k) < \infty\}$ and $\omega_x = \exp(2\pi\imath/x)$ ($x \ge 1$). Then the following hold for any $y \in I_\gamma$ and $\nu = 0, 1, \dots, d-1$.*

(a) $\delta(\mathbf{\Gamma}^*(y\omega_d^\nu)) = \delta(\mathbf{\Gamma}^*(y))$*, both of which are simple eigenvalues.*

(b) $\boldsymbol{\gamma}(y\omega_d^\nu) = \boldsymbol{\gamma}(y)\mathbf{\Delta}(\omega_d^\nu)^{-1}$ *and* $\boldsymbol{g}(y\omega_d^\nu) = \mathbf{\Delta}(\omega_d^\nu)\boldsymbol{g}(y)$.

[‡5] corrected: "an injective function" $\longrightarrow$ "a function"

[‡6] The published version states that function q is injective. However, this is not true, in general.

Proof. It follows from (67) that for $\nu = 0, 1, \dots, d-1$,

$$\begin{aligned}\boldsymbol{\Gamma}^*(y\omega_d^\nu) &= \boldsymbol{\Delta}(y\omega_d^\nu)\boldsymbol{\Lambda}^*(y^d)\boldsymbol{\Delta}(y\omega_d^\nu)^{-1}, \\ &= \boldsymbol{\Delta}(\omega_d^\nu)[\boldsymbol{\Delta}(y)\boldsymbol{\Lambda}^*(y^d)\boldsymbol{\Delta}(y)^{-1}]\boldsymbol{\Delta}(\omega_d^\nu)^{-1} \\ &= \boldsymbol{\Delta}(\omega_d^\nu)\boldsymbol{\Gamma}^*(y)\boldsymbol{\Delta}(\omega_d^\nu)^{-1}, \end{aligned} \tag{69}$$

which implies the statement (a) because $\boldsymbol{\Gamma}^*(y)$ is nonnegative and irreducible. Next we prove the statement (b). Pre-multiplying both sides of (69) by $\boldsymbol{\gamma}(y)\boldsymbol{\Delta}(\omega_d^\nu)^{-1}$ and using $\delta(\boldsymbol{\Gamma}^*(y\omega_d^\nu)) = \delta(\boldsymbol{\Gamma}^*(y))$, we have

$$\begin{aligned}\left[\boldsymbol{\gamma}(y)\boldsymbol{\Delta}(\omega_d^\nu)^{-1}\right]\boldsymbol{\Gamma}^*(y\omega_d^\nu) &= \delta(\boldsymbol{\Gamma}^*(y))\left[\boldsymbol{\gamma}(y)\boldsymbol{\Delta}(\omega_d^\nu)^{-1}\right] \\ &= \delta(\boldsymbol{\Gamma}^*(y\omega_d^\nu))\left[\boldsymbol{\gamma}(y)\boldsymbol{\Delta}(\omega_d^\nu)^{-1}\right]. \end{aligned} \tag{70}$$

Similarly we obtain

$$\boldsymbol{\Gamma}^*(y\omega_d^\nu)\left[\boldsymbol{\Delta}(\omega_d^\nu)\boldsymbol{g}(y)\right] = \delta(\boldsymbol{\Gamma}^*(y\omega_d^\nu))\left[\boldsymbol{\Delta}(\omega_d^\nu)\boldsymbol{g}(y)\right]. \tag{71}$$

It follows from (70) and (71) that there exist some constants φ_1 and φ_2 such that

$$\boldsymbol{\gamma}(y\omega_d^\nu) = \varphi_1\boldsymbol{\gamma}(y)\boldsymbol{\Delta}(\omega_d^\nu)^{-1}, \quad \boldsymbol{g}(y\omega_d^\nu) = \varphi_2\boldsymbol{\Delta}(\omega_d^\nu)\boldsymbol{g}(y).$$

We can easily confirm that $\varphi_1 = \varphi_2 = 1$ satisfies the normalizing condition (68). □

Theorem B.1 *Suppose Assumption B.1 holds and* $\delta(\boldsymbol{\Gamma}^*(y)) = 1$ *for some* $y \in I_\gamma$*, and let* ω *denote a complex number such that* $|\omega| = 1$*. Then* $\delta(\boldsymbol{\Gamma}^*(y\omega)) = 1$ *if and only if* $\omega^d = 1$*. Therefore*

$$d = \max\{n \in \mathbb{N}; \delta(\boldsymbol{\Gamma}^*(y\omega_n)) = 1\}. \tag{72}$$

Further if $\delta(\boldsymbol{\Gamma}^*(y\omega)) = 1$*, the eigenvalue is simple.*

Proof. Although Theorem B.1 can be proved in a similar way to Proposition 14 in [8], the proof is given in Appendix C.7 for completeness and the readers' convenience. □

Remark B.1 Theorem B.1 provides a definition of the period of MAdP $\{\boldsymbol{\Gamma}(k); k \in \mathbb{Z}\}$. In a very similar way, Shurenkov [23] defined the period of MAdPs with proper kernels. In the context of this paper, his definition is as follows:

$$d = \max\left\{n \in \mathbb{N}; \boldsymbol{\Gamma}^*(\omega_n)\boldsymbol{f} = \boldsymbol{f} \text{ for some } \boldsymbol{f} \in \mathbb{C}^J \text{ s.t. } |[\boldsymbol{f}]_j| = 1\ (j \in \mathbb{J})\right\}. \tag{73}$$

We can confirm that (73) is equivalent to Theorem B.1 if $\boldsymbol{\Gamma}^*(1)$ is stochastic. Shurenkov [23] also implied that the statement of Lemma B.2 holds, based on which Alsmeyer [2] defined the period of MAdPs.

C Proofs

C.1 Proof of Proposition 2.1

In order to prove this proposition by the reduction to absurdity, we assume the negation of the statement, i.e., either (i) $\boldsymbol{G}$ is strictly lower triangular, or (ii) $\boldsymbol{G}$ takes a form such that

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{G}_1 & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{G}_{2,1} & \boldsymbol{G}_2 & \boldsymbol{O} \\ \boldsymbol{G}_{3,1} & \boldsymbol{G}_{3,2} & \boldsymbol{G}_3 \end{pmatrix}, \tag{74}$$

where $\boldsymbol{G}_i$ $(i = 1, 2)$ is irreducible and $\boldsymbol{G}_2$ may be equal to $\boldsymbol{G}_\bullet$ (in that case, $\boldsymbol{G}_\bullet$ is irreducible). If case (i) is true, then $\boldsymbol{G}$ is a nilpotent matrix and thus $\boldsymbol{G}^m = \boldsymbol{O}$ for some $m \in \mathbb{N}$, which is inconsistent with Assumption 1.1 (a).

In what follows, we consider case (ii). For simplicity, we denote $\{(k, j); j \in \mathbb{M}\}$ by $\mathbb{L}(k)$ $(k \in \mathbb{N})$, and then partition $\mathbb{L}(k)$ into subsets $\mathbb{L}_1(k)$, $\mathbb{L}_2(k)$ and $\mathbb{L}_3(k)$ corresponding to $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ and $\boldsymbol{G}_3$, respectively. According to (74) and the definition of $\boldsymbol{G}$, for any $k \ge 2$ and l $(1 \le l < k)$ there exists no path from $\mathbb{L}_1(k)$ to $\mathbb{L}_2(l)$ avoiding $\mathbb{L}(0) \triangleq \{(0, j); j \in \mathbb{M}_0\}$.

We now fix $k \ge M+1$ (≥ 2). Since the cardinality of $\mathbb{M}$ is equal to M, it follows from Assumption 1.1 (b) that there exists a path from $\mathbb{L}_1(k)$ to $\cup_{m=0}^{\infty}\mathbb{L}_2(m)$ of length at most M. Such a path does not go through any state of $\mathbb{L}(0)$ because of the skip-free-to-the-left property of $\boldsymbol{T}$. Thus for some $l' \ge 1$, there exists a path from $\mathbb{L}_1(k)$ to $\mathbb{L}_2(l')$ avoiding $\mathbb{L}(0)$. If $l' < k$, we immediately have a contradiction. In fact, for $l' \ge k$, we also have a contradiction because there exists a path from any state of $\mathbb{L}_2(l')$ to any state of $\mathbb{L}_2(1)$ avoiding $\mathbb{L}(0)$, which follows from the irreducibility of $\boldsymbol{G}_2$. $\square$

C.2 Proof of Proposition 2.4

We prove this proposition by reduction to absurdity. The negation of the statement is that either (i) $\boldsymbol{R}$ is strictly upper triangular or (ii) $\boldsymbol{R}$ takes a form such that

$$\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_{1,2} & \boldsymbol{R}_{1,3} \\ \boldsymbol{O} & \boldsymbol{R}_2 & \boldsymbol{R}_{2,3} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{R}_3 \end{pmatrix}, \tag{75}$$

where $\boldsymbol{R}_i$ $(i = 1, 2)$ is irreducible and $\boldsymbol{R}_2$ may be equal to $\boldsymbol{R}_\bullet$. If case (i) is true, $\boldsymbol{R}$ is a nilpotent matrix, which contradicts (19).

Next we consider case (ii). We partition $\mathbb{L}(k)$ into subsets $\mathbb{L}_1(k)$, $\mathbb{L}_2(k)$ and

$\mathbb{L}_3(k)$ corresponding to $\boldsymbol{R}_1$, $\boldsymbol{R}_2$ and $\boldsymbol{R}_3$, respectively. Note here that[‡7]

$$[\boldsymbol{R}]_{i,j} = \mathrm{E}\left[\sum_{k=1}^{\infty} \sum_{n=1}^{a(k)-1} 1(X_n = k+1,\ S_n = j) \,\middle|\, X_0 = 1, S_0 = i \right],$$

where $1(\chi)$ denotes the indicator function of event χ. Thus (75) implies that for any $l \ge 2$ there exists no path from $\mathbb{L}_2(1)$ to $\mathbb{L}_1(l)$ avoiding $\mathbb{L}(1)$. On the other hand, owing to the irreducibility of $\boldsymbol{R}_2$, there exists some integer $k_* \ge M+2$ such that $\mathbb{L}_2(k_*)$ is reachable from $\mathbb{L}_2(1)$ avoiding $\mathbb{L}(1)$. Further for some $l_* \ge 2$, there exists a path from $\mathbb{L}_2(k_*)$ to $\mathbb{L}_1(l_*)$ avoiding $\mathbb{L}(1)$, because the cardinality of $\mathbb{M}$ is equal to M, $\boldsymbol{A}$ is irreducible and $\boldsymbol{T}$ has the skip-free-to-the-left property. Therefore we have a path from $\mathbb{L}_2(1)$ to $\mathbb{L}_1(l_*)$ via $\mathbb{L}_2(k_*)$ ($l_*, k_* \ge 2$) avoiding $\mathbb{L}(1)$. This yields a contradiction. □

C.3 Proof of Proposition 2.8

We suppose that there exists some $i \in \mathbb{M}$ such that $(0,i) \to (k,i)$ only for $k=0$. Since $\sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_A(k) = \boldsymbol{A}$ is irreducible (see Assumption 1.1 (b)), for each $j \in \mathbb{M}$ there exists a unique pair $(k_{i,j}, k_{j,i})$ such that $k_{i,j} + k_{j,i} = 0$ and

$$(0,i) \to (k_{i,j}, j), \qquad (0,j) \to (k_{j,i}, i).$$

It thus follows that for any $k_0 \in \mathbb{Z}$,

$$\Pr[\breve{X}_n \ge k_0 + K_{\min}\ (\forall n = 1,2,\dots) \mid \breve{X}_0 = k_0, \breve{S}_0 = i] = 1, \tag{76}$$

where $K_{\min} = \min_{j\in\mathbb{M}} k_{i,j}$. We now fix k_0 to be $k_0 = \max(1, 1-K_{\min})$. Clearly $k_0 \ge 1$ and $k_0 + K_{\min} = \max(1, 1+K_{\min}) \ge 1$. Therefore (76) yields

$$\Pr[\breve{X}_n \ge 1\ (\forall n = 1,2,\dots) \mid \breve{X}_0 = k_0, \breve{S}_0 = i] = 1. \tag{77}$$

As a result, from (25) and (77), we have

$$\Pr[X_n \ge 1\ (\forall n = 1,2,\dots) \mid X_0 = k_0, S_0 = i] = 1,$$

which contradicts Assumption 1.1 (a). □

C.4 Proof of Lemma 3.2

It follows from the definition of $r_m(z)$'s ($m = 1,2,\dots,M$) that

$$\mathrm{adj}(x\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z))(x\boldsymbol{I} - \boldsymbol{\Gamma}_A^*(z)) = \prod_{m=1}^{M} (x - r_m(z))\boldsymbol{I}. \tag{78}$$

[‡7] The equation of $[\boldsymbol{R}]_{i,j}$ is corrected.

Since $r_1(\theta\omega_\tau^\nu) = \delta(\mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu)) = 1$ ($\nu = 0, 1, \dots, \tau - 1$), (78) leads to

$$\mathrm{adj}(\boldsymbol{I} - \mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu))(\boldsymbol{I} - \mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu)) = \boldsymbol{O}, \qquad \nu = 0, 1, \dots, \tau - 1. \tag{79}$$

On the other hand, differentiating both sides of (78) with respect to x yields

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{adj}(x\boldsymbol{I} - \mathbf{\Gamma}_A^*(z))\right](x\boldsymbol{I} - \mathbf{\Gamma}_A^*(z)) + \mathrm{adj}(x\boldsymbol{I} - \mathbf{\Gamma}_A^*(z)) = \sum_{l=1}^{M} \prod_{m \in \mathbb{M}\setminus\{l\}} (x - r_m(z))\boldsymbol{I}.$$

Post-multiplying both sides of the above equation by $\boldsymbol{v}(z)\boldsymbol{\mu}(z)$, we obtain

$$\begin{aligned} &\left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{adj}(x\boldsymbol{I} - \mathbf{\Gamma}_A^*(z))\right][x - r_1(z)]\boldsymbol{v}(z)\boldsymbol{\mu}(z) + \mathrm{adj}(x\boldsymbol{I} - \mathbf{\Gamma}_A^*(z))\boldsymbol{v}(z)\boldsymbol{\mu}(z) \\ &\quad = \sum_{l=1}^{M} \prod_{m \in \mathbb{M}\setminus\{l\}} (x - r_m(z))\boldsymbol{v}(z)\boldsymbol{\mu}(z). \end{aligned} \tag{80}$$

Since $r_1(\theta\omega_\tau^\nu) = 1$ and $\prod_{m=2}^{M}(1 - r_m(\theta\omega_\tau^\nu)) \neq 0$, letting $x = 1$ and $z = \theta\omega_\tau^\nu$ in (80) yields

$$\mathrm{adj}(\boldsymbol{I} - \mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu))\boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu) = \prod_{m=2}^{M}(1 - r_m(\theta\omega_\tau^\nu))\boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu),$$

from which and (79) it follows that

$$\begin{aligned} &\mathrm{adj}(\boldsymbol{I} - \mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu))[\boldsymbol{I} - \mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu) + \boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu)] \\ &\quad = \prod_{m=2}^{M}(1 - r_m(\theta\omega_\tau^\nu))\boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu). \end{aligned} \tag{81}$$

Note here that $\boldsymbol{I} - \mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu) + \boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu)$ is non-singular and

$$\boldsymbol{\mu}(\theta\omega_\tau^\nu)\left[\boldsymbol{I} - \mathbf{\Gamma}_A^*(\theta\omega_\tau^\nu) + \boldsymbol{v}(\theta\omega_\tau^\nu)\boldsymbol{\mu}(\theta\omega_\tau^\nu)\right]^{-1} = \boldsymbol{\mu}(\theta\omega_\tau^\nu).$$

Thus (81) leads to (38). □

C.5 Proof of Theorem A.1

Statement (a). It follows from Assumption A.1 that there exists some $R > \sigma$ such that $f(z)$ is holomorphic in the domain $\{z \in \mathbb{C}; \sigma < |z| \le R\}$. We can choose P positive numbers r_j's ($j = 0, 1, \dots, P-1$) such that all the $C(\sigma_j, r_j)$'s are strictly inside $C(0, R)$ and any two of them have no intersection. Let $\mathbb{D}$ denote

$$\mathbb{D} = \{z; |z| < R\} \Big\backslash \bigcup_{j=0}^{P-1}\{z; |z - \sigma_j| \le r_j\}.$$

Clearly $f(z)$ is holomorphic in domain $\mathbb{D} \cup C(0,R) \cup C(\sigma_0, r_0) \cup \cdots \cup C(\sigma_{P-1}, r_{P-1})$. Thus by the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi\iota} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - \frac{1}{2\pi\iota} \sum_{j=0}^{P-1} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \qquad z \in \mathbb{D}, \tag{82}$$

where the integrals are taken counter-clockwise.

We now consider the first term in (82). For any $z \in \mathbb{D}$ and $\zeta \in C(0,R)$, we have $|z/\zeta| < 1$ and therefore

$$\frac{1}{2\pi\iota} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi\iota} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \frac{z^n}{\zeta^n} \mathrm{d}\zeta, \qquad z \in \mathbb{D}. \tag{83}$$

Since $f(\zeta)$ is holomorphic for $\zeta \in C(0,R)$, there exists some $f_{\max} > 0$ such that

$$|f(\zeta)| \leq f_{\max}, \qquad \zeta \in C(0,R). \tag{84}$$

Thus for any fixed $z \in \mathbb{D}$,

$$\left| \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \frac{z^n}{\zeta^n} \right| \leq \frac{f_{\max}}{R} \sum_{n=0}^{\infty} \left| \frac{z}{R} \right|^n = \frac{f_{\max}}{R} \frac{1}{1 - \left| \dfrac{z}{R} \right|} < \infty, \qquad \zeta \in C(0,R),$$

which shows that the order of summation and integration on the right hand side of (83) is interchangeable. As a result, it follows from (83) that

$$\frac{1}{2\pi\iota} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi\iota} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta \right) z^n = \sum_{n=0}^{\infty} c_n z^n, \tag{85}$$

where

$$c_n = \frac{1}{2\pi\iota} \oint_{C(0,R)} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta, \qquad n = 0, 1, \ldots. \tag{86}$$

Next we consider the second term in (82). Since $|\zeta - \sigma_j|/|z - \sigma_j| < 1$ for any $z \in \mathbb{D}$ and $\zeta \in C(\sigma_j, r_j)$,

$$\frac{1}{\zeta - z} = \frac{1}{(\sigma_j - z) - (\sigma_j - \zeta)} = \frac{1}{\sigma_j - z} \cdot \frac{1}{1 - \dfrac{\sigma_j - \zeta}{\sigma_j - z}} = \frac{1}{\sigma_j - z} \sum_{n=0}^{\infty} \left(\frac{\sigma_j - \zeta}{\sigma_j - z} \right)^n.$$

Thus we have

$$\frac{1}{2\pi\iota} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi\iota} \oint_{C(\sigma_j, r_j)} \sum_{n=1}^{\infty} \frac{f(\zeta)(\sigma_j - \zeta)^{n-1}}{(\sigma_j - z)^n} \mathrm{d}\zeta, \qquad z \in \mathbb{D}.$$

In a way very similar to the right hand side of (83), we can confirm that the order of summation and integration in the above equation is interchangeable, and then obtain

$$\frac{1}{2\pi\iota}\oint_{C(\sigma_j,r_j)}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{2\pi\iota}\oint_{C(\sigma_j,r_j)}f(\zeta)(\zeta-\sigma_j)^{n-1}\mathrm{d}\zeta\right)\cdot\frac{1}{(\sigma_j-z)^n},\qquad z\in\mathbb{D}. \tag{87}$$

Since $z=\sigma_j$ is an $\breve{m}$th order pole,

$$\frac{1}{2\pi\iota}\oint_{C(\sigma_j,r_j)}f(\zeta)(\zeta-\sigma_j)^{n-1}\mathrm{d}\zeta=0,\qquad \text{for all } n=\breve{m}+1,\breve{m}+2,\dots,$$

from which and (87) we have

$$\begin{aligned}
&\frac{1}{2\pi\iota}\oint_{C(\sigma_j,r_j)}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta\\
&\quad=\sum_{n=1}^{\breve{m}}(-1)^{n-1}\left(\frac{1}{2\pi\iota}\oint_{C(\sigma_j,r_j)}f(\zeta)(\zeta-\sigma_j)^{n-1}\mathrm{d}\zeta\right)\frac{1}{(\sigma_j-z)^n}\\
&\quad=-\sum_{n=1}^{\breve{m}}\sigma_j^{\breve{m}}\left[(-1)^{n+\breve{m}}\left(\frac{1}{2\pi\iota}\oint_{C(\sigma_j,r_j)}\frac{f(\zeta)(1-\zeta/\sigma_j)^{\breve{m}}}{(\zeta-\sigma_j)^{\breve{m}-n+1}}\mathrm{d}\zeta\right)\right]\frac{1}{(\sigma_j-z)^n}\\
&\quad=-\sum_{n=1}^{\breve{m}}\frac{\sigma_j^{\breve{m}}c_{j,n}}{(\sigma_j-z)^n},
\end{aligned}\tag{88}$$

where

$$c_{j,n}=(-1)^{n+\breve{m}}\cdot\frac{1}{2\pi\iota}\oint_{C(\sigma_j,r_j)}\frac{f(\zeta)(1-\zeta/\sigma_j)^{\breve{m}}}{(\zeta-\sigma_j)^{\breve{m}-n+1}}\mathrm{d}\zeta. \tag{89}$$

Substituting (85) and (88) into (82), we have

$$f(z)=\sum_{n=0}^{\infty}c_nz^n+\sum_{j=0}^{P-1}\sum_{n=1}^{\breve{m}}\frac{\sigma_j^{\breve{m}}c_{j,n}}{(\sigma_j-z)^n},\qquad z\in\mathbb{D},$$

and therefore

$$\sum_{n=0}^{\infty}x_nz^n=\sum_{n=0}^{\infty}c_nz^n+\sum_{j=0}^{P-1}\sum_{n=1}^{\breve{m}}\frac{\sigma_j^{\breve{m}}c_{j,n}}{(\sigma_j-z)^n},\qquad z\in\mathbb{D}\cap\{z\in\mathbb{C};|z|<\sigma\}. \tag{90}$$

Differentiating both sides of (90) k times with respect to z, dividing them by $k!$ and letting $z=0$ yield

$$x_k=c_k+\sum_{j=0}^{P-1}\sum_{n=1}^{\breve{m}}\sigma_j^{\breve{m}-n}c_{j,n}\binom{k+n-1}{n-1}\frac{1}{\sigma_j^k}. \tag{91}$$

It follows from (84) and (86) that

$$|c_k| \le \frac{1}{2\pi} \oint_{C(0,R)} \left| \frac{f(\zeta)}{\zeta^{k+1}} \right| \mathrm{d}\zeta \le \frac{1}{2\pi} \oint_{C(0,R)} \frac{f_{\max}}{R^{k+1}} \mathrm{d}\zeta = \frac{f_{\max}}{R^k},$$

which leads to

$$\lim_{k\to\infty} \left| \frac{c_k}{\dfrac{1}{\sigma_j^k}} \right| = \lim_{k\to\infty} |c_k|\sigma^k \le \lim_{k\to\infty} f_{\max} \left(\frac{\sigma}{R}\right)^k = 0, \qquad \text{for all } j = 0, 1, \dots, P-1, \tag{92}$$

where we use $|\sigma_j| = \sigma$ ($j = 0, 1, \dots, P-1$) and $0 < \sigma/R < 1$. From (91) and (92), we have

$$\begin{aligned} x_k &= \binom{k + \breve{m} - 1}{\breve{m} - 1} \frac{1}{\sigma^k} \sum_{j=0}^{P-1} \left(\frac{\sigma}{\sigma_j}\right)^k c_{j,\breve{m}} + O(a_{\breve{m},k}) \\ &= \frac{k^{\breve{m}-1}}{(\breve{m}-1)!} \frac{1}{\sigma^k} \sum_{j=0}^{P-1} \left(\frac{\sigma}{\sigma_j}\right)^k c_{j,\breve{m}} + O(a_{\breve{m},k}). \end{aligned} \tag{93}$$

Note here that (89) yields

$$c_{j,\breve{m}} = \frac{1}{2\pi\iota} \oint_{C(\sigma_j, r_j)} \frac{f(\zeta)(1 - \zeta/\sigma_j)^{\breve{m}}}{\zeta - \sigma_j} \mathrm{d}\zeta = \lim_{\zeta\to\sigma_j} \left(1 - \frac{\zeta}{\sigma_j}\right)^{\breve{m}} f(\zeta), \tag{94}$$

where we use the Cauchy integral formula in the last equality. As a result, the statement (a) is true.

Statement (b). From (64) and the definition of $\{a_{\breve{m},k}\}$, we have

$$x_k = \frac{k^{\breve{m}-1}}{(\breve{m}-1)!} \frac{1}{\sigma^k} \xi_k + o\left(\frac{k^{\breve{m}-1}}{\sigma^k}\right). \tag{95}$$

We now suppose $\limsup_{k\to\infty} \xi_k \le 0$. Then (95) yields

$$\limsup_{k\to\infty} \frac{x_k}{k^{\breve{m}-1}\sigma^{-k}} = 0,$$

which implies that for any $\varepsilon > 0$ there exists some positive integer $K_\varepsilon \ge \breve{m} - 1$ such that $x_k < \varepsilon(k^{\breve{m}-1}/\sigma^k)$ for all $k = K_\varepsilon, K_\varepsilon + 1, \dots$. Thus we have

$$f(y) \le \sum_{k=0}^{K_\varepsilon - 1} y^k x_k + \varepsilon \sum_{k=K_\varepsilon}^{\infty} k^{\breve{m}-1} \left(\frac{y}{\sigma}\right)^k, \qquad 0 \le y < \sigma. \tag{96}$$

Note that for $l = 1, 2, \dots$,

$$\sum_{k=l}^{\infty} k(k-1)\cdots(k-l+1) \left(\frac{y}{\sigma}\right)^k = (-1)^{l+1} l! \frac{\sigma y^l}{(y-\sigma)^{l+1}}. \tag{97}$$

Note also that there exists an $(\breve{m}-1)$-tuple $(b_1, b_2, \dots, b_{\breve{m}-1})$ of real numbers such that

$$k^{\breve{m}-1} = \sum_{l=1}^{\breve{m}-1} b_l \cdot k(k-1)\cdots(k-l+1). \tag{98}$$

It follows from (96), (97) and (98) that for any $\varepsilon > 0$,

$$0 \le \limsup_{y\uparrow\sigma} \left(1 - \frac{y}{\sigma}\right)^{\breve{m}} f(y) \le \varepsilon b_{\breve{m}-1}(\breve{m}-1)!.$$

Letting $\varepsilon \to 0$ in the above inequality, we have $\lim_{y\uparrow\sigma} \{1-(y/\sigma)\}^{\breve{m}} f(y) = 0$, which is inconsistent with Assumption A.1.

Statement (c). It follows from (94), Assumption A.2 and Remark A.1 that $c_{0,\breve{m}}$ is a real number and $(c_{j,\breve{m}}, c_{P-j,\breve{m}})$ $(j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor)$ is a pair of complex conjugates, and thus ξ_k is a real number such that

$$\xi_k = y_0 + \sum_{j=1}^{\lfloor (P-1)/2 \rfloor} y_j \cos(2\pi k \alpha_j), \qquad k = 0, 1, \dots, \tag{99}$$

where $y_j \in \mathbb{R}$ $(j = 0, 1, \dots, \lfloor (P-1)/2 \rfloor)$ and $0 \le \alpha_j < 1$ $(j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor)$.

In what follows, we assume $\xi_{k_0} < 0$ for some nonnegative integer k_0 and then prove the following.

Claim: There exists some $b > 0$ such that $\xi_k < -b$ for infinitely many k's.

If this is true, (95) implies that $x_k < 0$ for a sufficiently large k, which contradicts the fact that $x_k \ge 0$ for all $k = 0, 1, \dots$. As a result, for all $k = 0, 1, \dots$, ξ_k must be nonnegative, i.e., the statement (c) is true.

We split $\mathcal{A} \triangleq \{\alpha_j; j = 1, 2, \dots, \lfloor (P-1)/2 \rfloor\}$ into rational numbers and irrational numbers. We then define $\mathcal{A}_0$ as the set of the rational numbers of $\mathcal{A}$. Next we choose an irrational number α_{j_1} from $\mathcal{A}\backslash\mathcal{A}_0$ (if any) and let $\mathcal{A}_1 = \{\alpha_j \in \mathcal{A}\backslash\mathcal{A}_0; \alpha_j/\alpha_{j_1} \text{ is rational}\}$. Further we choose an irrational number α_{j_2} from $\mathcal{A}\backslash(\mathcal{A}_0 \cup \mathcal{A}_1)$ (if any) and let $\mathcal{A}_2 = \{\alpha_j \in \mathcal{A}\backslash(\mathcal{A}_0 \cup \mathcal{A}_1); \alpha_j/\alpha_{j_2} \text{ is rational}\}$. Repeating this procedure, we can obtain $\tilde{P}$ sets, $\mathcal{A}_j$'s $(j = 1, 2, \dots, \tilde{P})$, where $\tilde{P}$ may be equal to zero, i.e., all members of $\mathcal{A}$ may be rational. Let $\tilde{\alpha}_j$ $(j = 0, 1, \dots, \tilde{P})$ denote some number such that all members of $\mathcal{A}_j$ are multiples of $\tilde{\alpha}_j$. Then $\tilde{\alpha}_j$'s $(j = 0, 1, \dots, \tilde{P})$ are linearly independent over the rationals (see Definition D.1). Note here that for $n = 1, 2, \dots,$

$$\cos(nt) = T_n(\cos t), \qquad t \in \mathbb{R},$$

where $T_n(t)$'s $(n = 1, 2, \dots)$ denote the Chebyshev polynomials of the first kind. It thus follows from (99) that there exist some polynomial functions $\psi^{(\mathcal{A}_j)}$'s $(j = 0, 1, \dots, \tilde{P})$ on $\mathbb{R}$ such that

$$\xi_k = y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k\tilde{\alpha}_0) + \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi k\tilde{\alpha}_j), \qquad k = 0, 1, \dots, \tag{100}$$

where $\psi^{(\mathcal{A}_j)} \circ \cos(\cdot)$ denotes a composite function $\psi^{(\mathcal{A}_j)}(\cos(\cdot))$ of functions $\psi^{(\mathcal{A}_j)}(\cdot)$ and $\cos(\cdot)$. Since $\tilde{\alpha}_0$ is rational, there exists some $g \in \mathbb{N}$ such that

$$\psi^{(\mathcal{A}_0)} \circ (2\pi(ng+k)\tilde{\alpha}_0) = \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k\tilde{\alpha}_0), \qquad \text{for all } k, n = 0, 1, \dots. \tag{101}$$

Therefore in the case of $\tilde{P} = 0$, it follows from (100) and (101) that

$$\xi_{ng+k_0} = y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k_0\tilde{\alpha}_0) = \xi_{k_0} < 0, \qquad \text{for all } n = 0, 1, \dots,$$

which implies the above claim.

We next consider the case of $\tilde{P} \geq 1$. Since $g\tilde{\alpha}_1$, $g\tilde{\alpha}_2$, …, $g\tilde{\alpha}_{\tilde{P}}$ are linearly independent over the rationals, it follows from Proposition D.1 that for any $\varepsilon > 0$ and any $\boldsymbol{t} \triangleq (t_1, t_2, \dots, t_{\tilde{P}}) \in \mathbb{R}^{\tilde{P}}$, there exist integers $n_* := n_*(\varepsilon, \boldsymbol{t})$ and $l_j := l_j(\varepsilon, \boldsymbol{t})$ ($j = 1, 2, \dots, \tilde{P}$) such that

$$|(n_*g + k_0)\tilde{\alpha}_j - l_j - t_j| < \frac{\varepsilon}{2\pi}, \qquad j = 1, 2, \dots, \tilde{P}.$$

Thus since $\psi^{(\mathcal{A}_j)} \circ \cos(2\pi x)$ is a continuous function of x, there exists some $\delta := \delta(\varepsilon) > 0$ such that $\lim_{\varepsilon\downarrow 0} \delta = 0$ and

$$|\psi^{(\mathcal{A}_j)} \circ \cos(2\pi(n_*g + k_0)\tilde{\alpha}_j) - \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j)| < \delta, \qquad j = 1, 2, \dots, \tilde{P}. \tag{102}$$

It follows from (100), (101) and (102) that

$$\begin{aligned} &\left|\xi_{n_*g+k_0} - \left(y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k_0\tilde{\alpha}_0) + \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j)\right)\right| \\ &\qquad \leq \sum_{j=1}^{\tilde{P}} \left|\psi^{(\mathcal{A}_j)} \circ \cos(2\pi(n_*g + k_0)\tilde{\alpha}_j) - \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j)\right| < \tilde{P}\delta. \end{aligned} \tag{103}$$

We define $V_+(k)$ and $V_-(k)$ ($k = 0, 1, \dots$) as

$$V_+(k) = y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k\tilde{\alpha}_0) + \max_{\boldsymbol{t}\in\mathbb{R}^{\tilde{P}}} \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j),$$

$$V_-(k) = y_0 + \psi^{(\mathcal{A}_0)} \circ \cos(2\pi k\tilde{\alpha}_0) + \min_{\boldsymbol{t}\in\mathbb{R}^{\tilde{P}}} \sum_{j=1}^{\tilde{P}} \psi^{(\mathcal{A}_j)} \circ \cos(2\pi t_j),$$

respectively. It follows from the above definition and (100) that $V_-(k_0) \le \xi_{k_0} \le V_+(k_0)$. Further (103) implies that $\{\xi_{ng+k_0}; n = 0, 1, \dots\}$ is dense in the interval $[V_-(k_0), V_+(k_0)]$. Thus there exist infinitely many n's such that $\xi_{ng+k_0} < V_-(k_0)/2 < 0$. This completes the proof of the statement (c).

Statement (d). We prove this by reduction to absurdity, assuming $\xi_{\hat{k}} \le 0$ for some nonnegative integer $\hat{k}$. Since $(\arg \sigma_j)/\pi$ is a rational number for any $j = 0, 1, \dots, P-1$, there exist a positive integer g and nonnegative integers $l_0, l_1, \dots, l_{P-1}$ such $\sigma_j = \sigma \exp(\iota 2\pi l_j / g)$ ($j = 0, 1, \dots, P-1$). Clearly, $\xi_{ng+\hat{k}} \le 0$ for all $n = 0, 1, \dots$. It thus follows from (95) that for any $\varepsilon > 0$ there exists some nonnegative integer $\hat{n}$ such that $\hat{n}g + \hat{k} \ge \breve{m} - 1$ and

$$x_{ng+\hat{k}} \le \varepsilon \frac{(ng+\hat{k})^{\breve{m}-1}}{\sigma^{ng+\hat{k}}}, \qquad \text{for all } n = \hat{n}, \hat{n}+1, \dots.$$

We now fix $\hat{n}$ to be such that $\{x_k\}$ is nonincreasing for all $k \ge \hat{n}g + \hat{k}$ (recall that $\{x_k\}$ is eventually nonincreasing)[‡8]. It then follows that

$$x_{ng+\hat{k}+l} \le \varepsilon \frac{(ng+\hat{k})^{\breve{m}-1}}{\sigma^{ng+\hat{k}}}, \qquad n = \hat{n}, \hat{n}+1, \dots, \quad l = 0, 1, \dots, g-1,$$

which yields for $0 \le y < \sigma$,

$$\begin{aligned}
0 \le f(y) &\le \sum_{k=0}^{\hat{n}g+\hat{k}-1} y^k x_k + \varepsilon \sum_{n=\hat{n}}^{\infty} \frac{(ng+\hat{k})^{\breve{m}-1}}{\sigma^{ng+\hat{k}}} y^{ng+\hat{k}} \sum_{l=0}^{g-1} y^l \\
&\le C_1 + \varepsilon \frac{1-\sigma^g}{1-\sigma} \sum_{n=\hat{n}}^{\infty} (ng+\hat{k})^{\breve{m}-1} \left(\frac{y}{\sigma}\right)^{ng+\hat{k}} \\
&\le C_1 + \varepsilon C_2 \sum_{k=\breve{m}-1}^{\infty} k^{\breve{m}-1} \left(\frac{y}{\sigma}\right)^{k} \\
&\le C_1 + \varepsilon C_2 \sum_{k=\breve{m}-1}^{\infty} k^{\breve{m}-1} \left(\frac{y}{\sigma}\right)^{k-\breve{m}+1}, \qquad (104)
\end{aligned}$$

where $C_1 = \sum_{k=0}^{\hat{n}g+\hat{k}-1} \sigma^k x_k < \infty$ and $C_2 = (1-\sigma^g)/(1-\sigma)$. Note here that the second last inequality in (104) follows from $\hat{n}g + \hat{k} \ge \breve{m} - 1$ and the last one follows from $0 \le y/\sigma < 1$.

Let $\phi(y) = \sum_{k=0}^{\infty} (y/\sigma)^k = -\sigma(y-\sigma)^{-1}$ for $0 \le y < \sigma$. We then have for $0 \le y < \sigma$,

$$\frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} \phi(y) = \sum_{k=0}^{\infty} \frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}} \left(\frac{y}{\sigma}\right)^k = \frac{1}{\sigma^{\breve{m}-1}} \sum_{k=\breve{m}-1}^{\infty} k(k-1) \cdots (k-\breve{m}+2) \left(\frac{y}{\sigma}\right)^{k-\breve{m}+1}.$$

[‡8] This part is the only difference from the proof of the published version.

Thus for $1 \le l \le \breve{m} - 1$,

$$\sum_{k=\breve{m}-1}^{\infty} k(k-1)\cdots(k-l+1)\left(\frac{y}{\sigma}\right)^{k-\breve{m}+1} \le \sigma^{\breve{m}-1}\frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}}\phi(y).$$

Using this inequality and (98), we can bound $f(y)$ in (104) as follows.

$$0 \le f(y) \le C_1 + \varepsilon C \frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}}\phi(y),$$

where $C = C_2 \sigma^{\breve{m}-1} \sum_{l=1}^{\breve{m}-1} b_l$. Further,

$$\frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}}\phi(y) = -\sigma \frac{\mathrm{d}^{\breve{m}-1}}{\mathrm{d}y^{\breve{m}-1}}(y-\sigma)^{-1} = \sigma(-1)^{\breve{m}}(\breve{m}-1)!(y-\sigma)^{-\breve{m}}.$$

As a result,

$$0 \le \limsup_{y\uparrow\sigma}\left(1 - \frac{y}{\sigma}\right)^{\breve{m}} f(y) \le \varepsilon C(\breve{m}-1)!\sigma^{-\breve{m}+1}.$$

Letting $\varepsilon \to 0$ in the above inequality, we have $\lim_{y\uparrow\sigma}(1 - y/\sigma)^{\breve{m}} f(y) = 0$, which contradicts Assumption A.1. □

C.6 Proof of Lemma B.1

Under Assumption B.1, for any $i, j \in \mathbb{J}$ ($i \neq j$) there exist integers $k_{i,j}$ and $k_{j,i}$ ($k_{i,j} + k_{j,i} \neq 0$) such that $(0,i) \to (k_{i,j}, j)$ and $(0,j) \to (k_{j,i}, i)$. Let $\mathbb{K}_{j\to i\to j}$ denote

$$\mathbb{K}_{j\to i\to j} = \{k_{j,i} + k_{i,j}\} \cup \{k_{j,i} + k + k_{i,j};\ k \in \mathbb{K}_i\}. \tag{105}$$

Clearly $\mathbb{K}_{j\to i\to j} \subseteq \mathbb{K}_j$ and therefore

$$\gcd\{k \in \mathbb{K}_{j\to i\to j}\} \ge \gcd\{k \in \mathbb{K}_j\} = d_j. \tag{106}$$

In what follows, we prove $\gcd\{k \in \mathbb{K}_{j\to i\to j}\} \le d_i$, from which and (106) it follows that $d_j \le d_i$. Interchanging i and j in the proof of $d_j \le d_i$, we can readily show that $d_i \le d_j$. Therefore we have $d_i = d_j$.

Since $(0,i) \to (k_{i,j}, j) \to (k_{i,j}+k_{j,i}, i)$, we have $k_{i,j}+k_{j,i} \in \mathbb{K}_i$ and therefore $k_{i,j} + k_{j,i} = a_0 d_i$ for some integer $a_0 \neq 0$. Note here that $\mathbb{K}_i$ has at least two elements because

$$\{k_{i,j} + k_{j,i}\} \cup \{k_{i,j} + k + k_{j,i};\ k \in \mathbb{K}_j\} \subseteq \mathbb{K}_i.$$

Thus there exists a couple of nonzero integers (a_1, a_2) such that $\{a_1 d_i, a_2 d_i\} \subseteq \mathbb{K}_i$ and $\gcd\{a_1, a_2\} = 1$, due to $d_i = \gcd\{k \in \mathbb{K}_i\}$. It follows from (105) and $k_{i,j} + k_{j,i} = a_0 d_i$ that

$$\begin{aligned}
\mathbb{K}_{j\to i\to j} &\supseteq \{k_{j,i} + k_{i,j}\} \cup \{k_{j,i} + a_1 d_i + k_{i,j},\ k_{j,i} + a_2 d_i + k_{i,j}\} \\
&= \{a_0 d_i\} \cup \{a_0 d_i + a_1 d_i,\ a_0 d_i + a_2 d_i\} \\
&= \{a_0 d_i, (a_0 + a_1) d_i, (a_0 + a_2) d_i\},
\end{aligned}$$

which leads to $\gcd\{k \in \mathbb{K}_{j\to i\to j}\} \le \gcd\{a_0 d_i, (a_0 + a_1) d_i, (a_0 + a_2) d_i\} = d_i$. □

C.7 Proof of Theorem B.1

Since the if-part follows from Lemma B.3, we prove the only-if part. Let $\boldsymbol{V}(\omega)$ denote a $J \times J$ matrix such that

$$\boldsymbol{V}(\omega) = \mathrm{diag}(\boldsymbol{g}(y))^{-1}\boldsymbol{\Gamma}^*(y\omega)\mathrm{diag}(\boldsymbol{g}(y)), \qquad |\omega| = 1,$$

where $\mathrm{diag}(\boldsymbol{x})$ denotes a diagonal matrix whose jth diagonal element is equal to $[\boldsymbol{x}]_j$ for a vector $\boldsymbol{x}$. It is easy to see that $\boldsymbol{V}(1)$ is irreducible and stochastic and $\delta(\boldsymbol{V}(\omega)) = \delta(\boldsymbol{\Gamma}^*(y\omega)) = 1$. Let $\boldsymbol{f} = (f_j; j \in \mathbb{J})$ denote a right eigenvector of $\boldsymbol{V}(\omega)$ corresponding to $\delta(\boldsymbol{V}(\omega)) = 1$. We then have for any $n \in \mathbb{N}$, $(\boldsymbol{V}(\omega))^n \boldsymbol{f} = \boldsymbol{f}$ and thus

$$f_i = \sum_{j\in\mathbb{J}} [(\boldsymbol{V}(\omega))^n]_{i,j} f_j = \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{Z}} y^k [\boldsymbol{\Gamma}^{(n)}(k)]_{i,j} \frac{[\boldsymbol{g}(y)]_j}{[\boldsymbol{g}(y)]_i} \cdot \omega^k f_j, \qquad i,j \in \mathbb{J}, \tag{107}$$

where $\{\boldsymbol{\Gamma}^{(n)}(k); k \in \mathbb{Z}\}$ is the nth-fold convolution of $\{\boldsymbol{\Gamma}(k); k \in \mathbb{Z}\}$ with itself. Note that $\sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{Z}} y^k [\boldsymbol{\Gamma}^{(n)}(k)]_{i,j}[\boldsymbol{g}(y)]_j/[\boldsymbol{g}(y)]_i = 1$ because $(\boldsymbol{V}(1))^n \boldsymbol{e} = \boldsymbol{e}$. Let i' denote an element of $\mathbb{J}$ such that $|f_{i'}| \ge |f_j|$ for all $j \in \mathbb{J}$. It then follows from (107) that for any $j \in \mathbb{J}$,

$$\omega^k \frac{f_j}{f_{i'}} = 1 \text{ if } [\boldsymbol{\Gamma}^{(n)}(k)]_{i',j} > 0. \tag{108}$$

Since $\boldsymbol{\Gamma}^*(1)$ is irreducible, for any $j \in \mathbb{J}$ there exist some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $[\boldsymbol{\Gamma}^{(n)}(k)]_{i',j} > 0$. Thus (108) implies that $|f_j|$'s are all equal, because $|\omega| = 1$. We now consider a path from phase i to phase i such that

$$(0,i) \to (k_1, i_1) \to (k_2, i_2) \to (k_m, i_m) \triangleq (k_m, i),$$

where $(k_l, i_l) \in \mathbb{Z} \times \mathbb{J}$ for $l = 1, 2, \dots, m$ and $m \in \mathbb{N}$. Since the period of MAdP $\{\boldsymbol{\Gamma}(k)\}$ is equal to d, $k_1 + k_2 + \cdots + k_m$ is a multiple of d. From (108), we have $\omega^{k_1+k_2+\cdots+k_m} = 1$ and thus $\omega^d = 1$. The proof of the only-if part is completed.

As for the remaining statements, (72) is obvious, and it follows from Lemma B.3 (a) that if $\delta(\boldsymbol{\Gamma}^*(y\omega)) = 1$, then the eigenvalue $\delta(\boldsymbol{\Gamma}^*(y\omega_d^\nu)) = 1$ is simple for $\nu = 0, 1, \dots, d-1$. □

D Kronecker's Approximation Theorem

The following is Kronecker's approximation theorem. For details, see, e.g., Theorem 7.10 in [3].

Proposition D.1 *Let γ_i's ($i = 1, 2, \dots, n$) denote arbitrary real numbers. Let β_i's ($i = 1, 2, \dots, n$) denote arbitrary real numbers such that $\beta_1, \beta_2, \dots, \beta_n$ and*

1 are linearly independent over the rationals (see Definition D.1 below). Then for any $\varepsilon > 0$*, there exist an* $(n+1)$*-tuple* $(k, l_1, l_2, \ldots, l_n)$ *of integers such that*

$$|k\beta_i - l_i - \gamma_i| < \varepsilon, \qquad \textit{for all } i = 1, 2, \ldots, n, \tag{109}$$

and thus for any $\varepsilon > 0$ *and any* $\tilde{\gamma}_i \in [0, 1]$*,*

$$|k\beta_i - \lfloor k\beta_i \rfloor - \tilde{\gamma}_i| < \varepsilon, \qquad \textit{for all } i = 1, 2, \ldots, n,$$

which implies that $k\beta_i - \lfloor k\beta_i \rfloor$ $(k \in \mathbb{Z})$ *is dense in the interval* $[0, 1]$.

Definition D.1 Arbitrary real numbers β_i's $(i = 1, 2, \ldots, n)$ are said to be *linearly independent over the rationals (equivalently integers)* if there exists no set of rational numbers q_i's $(i = 1, 2, \ldots, n)$ such that $(q_1, q_2, \ldots, q_n) \neq \mathbf{0}$ and

$$\beta_1 q_1 + \beta_2 q_2 + \cdots + \beta_n q_n = 0. \tag{110}$$

Therefore if β_i's are linearly independent over the rationals, (110) implies that $q_1 = q_2 = \cdots = q_n = 0$.

E Example against Assumption 1.3

We suppose $\boldsymbol{A}(k)$'s ($k = 0, 1, \ldots$) are scalars such that for some finite $r > 1$,

$$\boldsymbol{A}(k) = \frac{1}{r^k} \frac{1}{(k+1)^3} \Big/ \sum_{n=0}^{\infty} \frac{1}{r^n} \frac{1}{(n+1)^3}, \qquad k = 0, 1, \ldots.$$

Clearly, $r_A = r$. We define $F(x)$ ($x \geq 1$) as

$$F(x) = x \sum_{k=0}^{\infty} \frac{1}{x^k} \frac{1}{(k+1)^3}.$$

It then follows that for any $x \geq 1$,

$$F'(x) = 1 - \sum_{k=2}^{\infty} \frac{1}{x^k} \frac{k-1}{(k+1)^3},$$

which leads to

$$\begin{aligned} &F'(1) \geq 1 - \sum_{k=2}^{\infty} \frac{1}{(k+1)^2} = \frac{9}{4} - \frac{\pi^2}{6} > 0, \\ &F''(x) > 0. \end{aligned}$$

Thus since $F(r_A) > F(1)$, we have

$$\frac{1}{r_A} \delta(\boldsymbol{A}^*(r_A)) = \frac{1}{r_A} \boldsymbol{A}^*(r_A) = \frac{F(1)}{F(r_A)} < 1.$$

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